THREE PARAMETERS FOR RELATING SMALL-SCALE TEMPORAL FADING STATISTICS TO MULTIPATH ANGLES-OF-ARRIVAL

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Abstract – The accurate classification of autocorrelation (second-order) small-scale fading statistics for mobile receivers over a local area is essential for studies involving equalization, diversity, adaptive arrays, and any other wireless concept affected by the time-varying channel or multipath angle-of-arrival characteristics. This paper establishes a relationship between small-scale fading and the directivity of impinging waves at a mobile receiver by deriving second-order fading statistics in terms of three new multipath angle-of-arrival parameters.

I. INTRODUCTION

The characterization of small-scale fading statistics is essential to the design of any mobile communications system [1]-[4]. The angles-of-arrival of impinging multipath power determine the second-order statistics of fading, i.e., how a fading signal evolves over time as a mobile receiver moves in space [5, 6, 7]. Classical analysis of second-order fading statistics usually assumes that multipath power arrives at the receiver from all directions, uniformly distributed about the azimuth [8]. However, recent research has shown that mobile wireless receivers operate in environments that bear little resemblance to the omnidirectional multipath model, particularly when a directional antenna is used at the receiver [9, 10]. Thus, a simple, comprehensive relationship between small-scale fading and multipath angle-of-arrival is valuable for design or deployment of a wireless mobile communication system where propagation is not omnidirectional.

The analysis in this paper is a powerful extension of the basic relationship presented in [11]. This paper provides a definition for fading rate variance — a measure of how fast power fluctuates for a mobile wireless receiver over a local area — and derives an equation for local area fading rate variance in terms of spatial channel parameters. Furthermore, these spatial parameters are shown to have basic physical meanings, related to the multipath angle-of-arrival characteristics. Three examples of directional multipath models are presented to clarify this relationship. The key result is Eqn (1), an expression relating fading behavior to multipath propagation of arbitrary fading complexity.

II. FADEING RATE VARIANCE

One of the best ways to gauge the effects of multipath angle-of-arrival on small-scale fading is to calculate the mean-square time rate-of-change of received power [11]. Since this value represents the variance of the time derivative of received power, we refer to this as the fading rate variance, denoted $\sigma_p^2$. Appendix A derives the basic fading rate variance equation in terms of three shape factors, $\Lambda$, $\gamma$, and $\theta_{\text{max}}$:

$$\sigma_p = \omega_{\text{max}} \Lambda P_T \sqrt{1 + \gamma \cos[2(\theta - \theta_{\text{max}})]}$$  \hspace{1cm} (1)

where $P_T$ is the average total power, $\theta$ is the azimuthal direction of receiver travel, and $\omega_{\text{max}}$ is the maximum possible Doppler frequency ($\omega_{\text{max}} = \frac{2v}{\lambda}$, where $v$ is the velocity of the receiver and $\lambda$ is the wavelength of the carrier frequency). This equation requires three spatial parameters that are based on the angular distribution of multipath power, $p(\theta)$:

$$\Lambda = \sqrt{1 - \frac{|F_1|^2}{|F_0|^2}}$$  \hspace{1cm} (2)

$$\gamma = \frac{|F_0 F_2 - F_1^2|}{|F_0|^2 - |F_1|^2}$$  \hspace{1cm} (3)

$$\theta_{\text{max}} = \frac{1}{2} \text{Phase}[F_0 F_2 - F_1^2]$$  \hspace{1cm} (4)
where

\[ F_n = \int_0^{2\pi} p(\theta) \exp(jn\theta) d\theta \]  

(5)

\( F_n \) is the \( n \)th complex Fourier coefficient of the azimuthal distribution of multipath power.

The parameters \( \Lambda, \gamma, \) and \( \theta_{\text{max}} \) represent some basic geometrical properties of multipath propagation. Furthermore, each parameter describes different aspects of local area fading behavior. To see this most clearly, Eqn (1) may be compared to the baseline case of fading in an omnidirectional channel, whose behavior is well understood [5, 8]. The fading rate variance for the omnidirectional fading channel, denoted as \( \sigma_{\text{omni}}^2 \), is given by 

\[ \sigma_{\text{omni}}^2 = \omega_{\text{omni}}^2 P^2 \]

which is independent of the direction of receiver travel [5, 11]. Therefore, the RMS fading rate relative to the omnidirectional fading rate is 

\[ \sigma = \frac{\sigma_p}{\sigma_{\text{omni}}} = \Lambda \sqrt{1 + \gamma \cos(2(\theta - \theta_{\text{max}}))} \]  

(6)

Eqn (6) predicts that the RMS fading rate for a local area will change as a function of the direction of receiver travel, \( \theta \), but will always fall within the range 

\[ \sqrt{1 - \gamma} \leq \frac{\sigma}{\Lambda} \leq \sqrt{1 + \gamma} \]  

(7)

Eqn (6) also predicts that, on average, the RMS fading rate is proportional to \( \Lambda \) [11].

Eqn (6) and Eqn (7) show the effects of multipath spatial parameters on the average fading rate. The physical meaning of each parameter is defined below and the specific influence on local area fading is discussed:

- The parameter \( \Lambda \) is a measure of angular spread. This parameter ranges between 0 and 1. Decreasing values of angular spread indicate that multipath power is becoming more concentrated about a single direction. The average fading in a local area decreases proportional to angular spread.

- The parameter \( \gamma \) is a measure of angular constriction. This parameter also ranges between 0 and 1 and describes how the average fading rate depends on the direction of receiver travel. Increasing values of angular constriction indicate that multipath power is becoming more concentrated about two directions. Eqn (1) shows that the range of possible average fading rates in a local area increases as angular constriction increases.

- The parameter \( \theta_{\text{max}} \) determines the azimuthal direction of maximum fading rate. This parameter is a companion to the constriction parameter.

It indicates the direction a narrowband receiver would travel to maximize the fading rate in a local area.

### III. EXAMPLES

The following three examples illustrate how to apply the definitions of angular spread and constriction to generic propagation problems to provide insight into the second order statistics of small-scale fading.

#### Two-Wave Model

Consider a simple, theoretical situation where two discrete multipath components, with individual powers defined by \( P_1 \) and \( P_2 \), arrive at a receiver separated by an azimuthal angle \( \alpha \). Figure 1 illustrates this angular distribution of power, which is mathematically defined as

\[ p(\theta) = P_1 \delta(\theta - \theta_o) + P_2 \delta(\theta - \theta_o - \alpha) \]  

(8)

where \( \theta_o \) is an arbitrary offset angle and \( \delta(\cdot) \) is a unit impulse. The expressions for \( \Lambda, \gamma, \) and \( \theta_{\text{max}} \) for this distribution are

\[ \Lambda = \frac{2\sqrt{P_1 P_2}}{P_1 + P_2} \sin \frac{\alpha}{2}, \quad \gamma = 1, \quad \theta_{\text{max}} = \theta_o + \frac{\alpha + \pi}{2} \]  

(9)

The value for \( \gamma \) is always 1 because the two-wave model represents perfect clustering about two directions (the physical definition of angular constriction). The limiting case of two multipath arriving from the same direction (\( \alpha = 0 \)) results in zero angular spread. An angular spread of 1 results only when two multipath of identical powers (\( P_1 = P_2 \)) are separated by \( \alpha = 180^\circ \). Figure 1 shows how the fading behavior changes as a function of multipath separation angle, \( \alpha \). As \( \alpha \) increases, a slowly fading channel becomes a more rapidly fading channel that exhibits a great deal of dependence on the receiver’s direction of travel.

#### Sector Propagation Model

Consider another theoretical situation where multipath power is arriving uniformly over a range of angles. The function \( p(\theta) \) will be defined by

\[ p(\theta) = \begin{cases} \frac{P_\alpha}{\alpha} & : \theta_o \leq \theta \leq \theta_o + \alpha \\ 0 & : \text{elsewhere} \end{cases} \]  

(10)

The angle \( \alpha \) indicates the width of the sector (in radians) of arriving multipath power and the angle \( \theta_o \) is an arbitrary offset angle, as illustrated by Figure 2. The expressions for \( \Lambda, \gamma, \) and \( \theta_{\text{max}} \) for this distribution are

\[ \Lambda = \sqrt{1 - \frac{4\sin^2 \frac{\alpha}{2}}{\alpha^2}}, \quad \gamma = \frac{4\sin^2 \frac{\alpha}{2} - \alpha \sin \alpha}{\alpha^2 - 4\sin^2 \frac{\alpha}{2}} \]

(11)
The limiting cases of these parameters and Eqn (1) provide deeper understanding of angular spread and constriction.

Figure 2 graphs the spatial channel parameters, $\Lambda$ and $\gamma$, as a function of sector width, $\alpha$. The limiting case of a single multipath arriving from precisely one direction corresponds to $\alpha = 0$, which results in the minimum angular spread of $\Lambda = 0$. The other limiting case of uniform illumination in all directions corresponds to $\alpha = 360^\circ$, which results in the maximum angular spread of $\Lambda = 1$. The angular constriction, $\gamma$, follows an opposite trend. It is at a maximum ($\gamma = 1$) when $\alpha = 0$ and at a minimum ($\gamma = 0$) when $\alpha = 360^\circ$. Figure 2 shows that as the multipath propagation is condensed into a smaller and smaller sector, the discrepancy between direction-dependent fading rates within the same local area increases. Overall, however, fading rates decrease with decreasing sector size $\alpha$.

Double-Sector Propagation Model

Another example of angular constriction effects may be studied using the spatial channel of Figure 3. Diffuse multipath propagation over two equal and opposite sectors of azimuthal angles characterizes the incoming power. The equation that describes this angular distribution of power is

$$p(\theta) = \left\{ \begin{array}{ll} \frac{\alpha}{2} & : \theta_o \leq (\theta \mod \pi) \leq \theta_o + \alpha \\ 0 & : \text{elsewhere} \end{array} \right. \quad (12)$$

Note that the value of angular spread, $\Lambda$, is always 1. Regardless of the value of $\alpha$, an equal amount of power arrives from opposite directions, producing no clear bias in the direction of power arrival.

The limiting case of $\alpha = 180^\circ$ (uniform propagation over all azimuthal angles) results in an angular constriction of $\gamma = 0$. This omnidirectional propagation produces fading statistics that are isotropic, invariant of the receiver’s direction of travel. As $\alpha$ decreases, the angular distribution of power becomes more and more constricted. In the limit of $\alpha = 0$, the value of angular constriction reaches its maximum, $\gamma = 1$. This case corresponds to the above-mentioned instance of two-wave propagation. Figure 3 shows how the fading behavior changes as sector width $\alpha$ increases, making the fading rate more and more isotropic while the RMS average remains constant.
IV. CONCLUSIONS

This paper has defined the fading rate variance, a measure of how rapidly the power of a continuous-wave signal fluctuates for a mobile receiver. The fading rate variance is a convenient and intuitive way of characterizing how rapidly a fading signal fluctuates. Furthermore, this paper developed exact relationships that relate the fading rate variance to the multipath angle-of-arrival characteristics of a local area. In fact, only three physically intuitive parameters - angular spread, angular constriction, and the direction of maximum fading - are required to exactly characterize the fading rate variance. These simple parameters dominate the second-order statistics of local area fading. The effects of additional multipath angle-of-arrival information are slight.

REFERENCES


A. DERIVATION OF FADING RATE VARIANCE

The power spectral density (PSD) of a baseband complex received voltage signal is related to the angular distribution of multipath power about the azimuth [6]:

\[
S_V(\omega) = \frac{p(\theta + \cos^{-1}\frac{\omega}{\omega_{\text{max}}}) + p(\theta - \cos^{-1}\frac{\omega}{\omega_{\text{max}}})}{\sqrt{\omega_{\text{max}}^2 - \omega^2}}
\]

(14)

where \(\theta\) is the azimuthal direction of travel and \(p(\cdot)\) is the angular distribution of impinging multipath power. The second moment of the fading process is given by the following integration [1, 12, 13]:

\[
\sigma^2_V = \omega_{\text{max}} \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} (\omega - \omega_c)^2 S_V(\omega) d\omega
\]

(15)

where \(\omega_c\) is the centroid of the PSD:

\[
\omega_c = \frac{1}{F_0} \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} \omega S_V(\omega) d\omega
\]

(16)

\(F_0\) is defined by Eqn (5) – this is really just the average total power of the fading process, \(P_T\).

Now insert Eqn (16) into Eqn (15), making the change of variable \(\theta_0 = \theta \pm \cos^{-1}\frac{\omega}{\omega_{\text{max}}},\) where the + and − signs correspond to the left and right terms of \(p(\cdot),\) respectively, of Eqn (14). After rearranging the limits of integration, the equation for \(\sigma^2_V\) becomes

\[
\sigma^2_V = \frac{F_0 \omega_{\text{max}}}{2} \left[ \frac{2\pi}{0} p(\theta_0) \cos(\theta - \theta_0) d\theta_0 \right]^2
\]

\[
+ \frac{\omega_{\text{max}}}{2} \left[ \frac{2\pi}{0} p(\theta_0) \cos[2(\theta - \theta_0)] d\theta_0 \right]
\]

(17)

Consider a complex Fourier expansion of \(\sigma^2_V\) with respect to \(\theta:\)

\[
\sigma^2_V = \text{Real} \left\{ \sum_{n=0}^{\infty} A_n \exp(-jn\theta) \right\}
\]

\[
= A_0 + \text{Real} \left\{ A_2 \exp(-j2\theta) \right\}
\]

(18)

All of the \(A_n\) are zero for odd \(n.\) This is because \(\sigma^2_V(\theta) = \sigma^2_V(\theta + \pi);\) that is, a 180° change in the direction of mobile travel should produce identical statistics. Furthermore, Eqn (17) has no harmonic content with respect to \(\theta\) for \(n > 2.\) Solving for the only two remaining complex coefficients produces

\[
A_0 = \frac{1}{2\pi} \int_{0}^{2\pi} \sigma^2_V d\theta = \frac{\omega_{\text{max}}^2}{2} \left[ F_0 - \frac{|F_1|^2}{F_0} \right]
\]

\[
= \frac{\omega_{\text{max}}^2}{2} \Lambda^2 P_T
\]

(19)

\[
A_2 = \frac{1}{\pi} \int_{0}^{2\pi} \sigma^2_V \exp(j2\theta) d\theta = \frac{\omega_{\text{max}}}{2} \left[ F_2 - \frac{F_1^2}{F_0} \right]
\]

\[
= A_0 \gamma \exp(j2\theta_{\text{max}})
\]

(20)

where \(\Lambda, \gamma,\) and \(\theta_{\text{max}}\) are the three basic spatial channel parameters defined in Eqn (2)-Eqn (4). If these two coefficients are placed back into Eqn (19), the end result is the relationship for \(\sigma^2_V:\)

\[
\sigma^2_V(\theta) = \frac{\omega_{\text{max}}^2 \Lambda^2 P_T}{2} \left( 1 + \gamma \cos[2(\theta - \theta_{\text{max}})] \right)
\]

(21)

The stochastic process of power is defined as \(P(t) = V^*(t)V(t)\) (units of volts squared). Thus, the PSD of power is the convolution of two complex voltage PSD’s: \(S_P(\omega) = S_V(\omega) \otimes S_V(-\omega).\) The rate variance relationship for power may then be written as

\[
\sigma^2_P = \int_{-\infty}^{\infty} \omega^2 S_P(\omega) d\omega
\]

(22)

Making the substitution \(\omega = \lambda - \omega'\) leads to

\[
\sigma^2_P = \int_{-\infty}^{\infty} (\lambda - \omega')^2 S_V(\lambda) S_V(\omega') d\omega'
\]

(23)

which may be regrouped and re-expressed in terms of the spectral centroid, \(\omega_c:\)

\[
\sigma^2_P = 2 \left[ \int_{-\infty}^{\infty} S_V(\omega) d\omega \right] \left[ \int_{-\infty}^{\infty} (\omega - \omega_c)^2 S_V(\omega) d\omega \right]
\]

(24)

Now substitute Eqn (21) for \(\sigma^2_V\) to obtain Eqn (1).