ESF1: Coulomb's Law in Cartesian Coordinates

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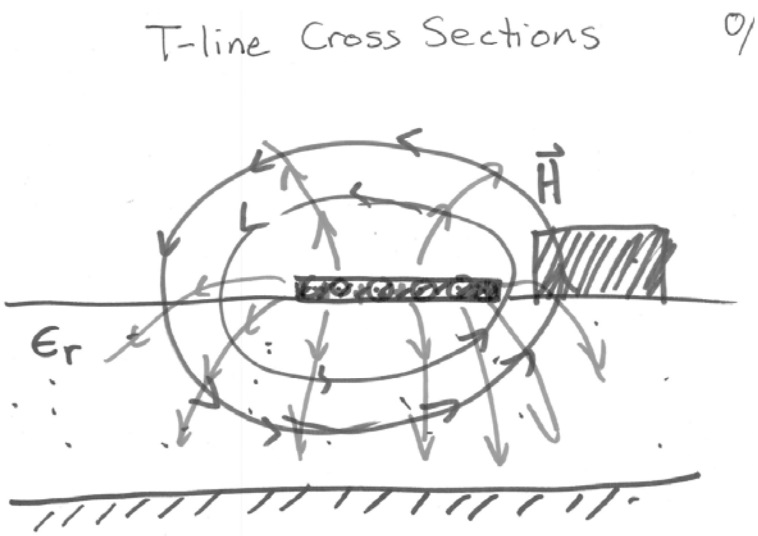
Field Theory Needed to Analyze Tlines



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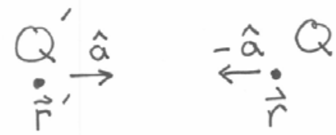
Coulomb's Law

From Physics, a system of Point Charges



$\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

units of meters

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

Cartesian unit vectors

Force in this system

$$\vec{F} = -\vec{F}\,' = \frac{Q'Q}{4\pi\epsilon R^2}\hat{a}$$

Definition of Electric Field

$$\vec{E}(\vec{r}) = \frac{\vec{F}_Q(\vec{r})}{Q}$$

Vector Formulation of Coulomb's Law

Distance between points:

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$= \|\vec{r} - \vec{r}\,'\|$$

Unit vector

$$\hat{a} = \frac{(\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|}$$

provides direction from Q' to Q

normalizes vector

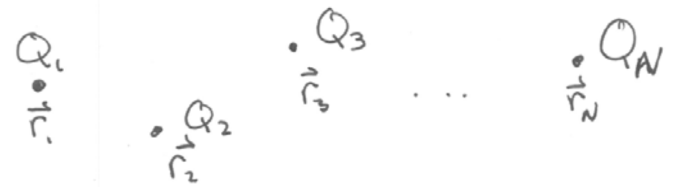
Thus

$$\vec{E}(\vec{r}) = \frac{Q'(\vec{r} - \vec{r}\,')}{4\pi\epsilon \|\vec{r} - \vec{r}\,'\|^3}$$

falls off with distance squared, need an extra $\|\vec{r} - \vec{r}\,'\|$ to normalize.

Superposition of Point Charges

Super Position of Point Charges



$$\vec{E}(\vec{r}) = \sum_{n=1}^{N} \frac{Q_n(\vec{r}-\vec{r}_n)}{4\pi\epsilon_0 \|\vec{r}-\vec{r}_n\|^3}$$

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Volume Charge Density

ρ_v - volume charge density (C/m^3)

total charge

variable of integration

$$Q = \int_{vol} \rho_v(\vec{r}\,')\, dv'$$

in Cartesian coordinates $= \int dx' \int dy' \int dz'\ \rho_v(x',y',z')$

or

$$= \iiint \rho_v(x',y',z')\, dx'\, dy'\, dz'$$

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Coulomb's Law for Volume Charges

Then

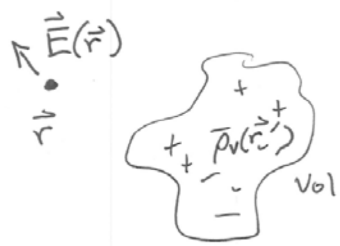
variables of integration

$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

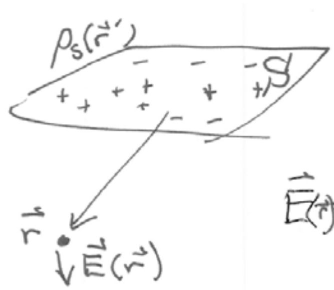
$$\vec{E}(\vec{r}) = \int_{vol} \frac{\rho_v(\vec{r}')\,dv'}{4\pi\epsilon_0 \|\vec{r}-\vec{r}'\|^3}$$

point of observation

$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$



Surface Charge Density

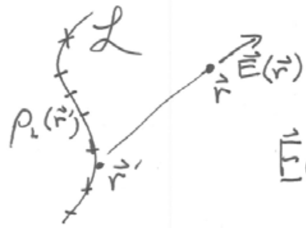


surface charge density $\rho_s(\vec{r}')$ C/m^2

$$\vec{E}(\vec{r}) = \int_S \frac{\rho_s(\vec{r}')\,ds'(\vec{r}-\vec{r}')}{4\pi\epsilon_0 \|\vec{r}-\vec{r}'\|^3}$$

$$Q = \int_S \rho_s(\vec{r}')\,ds'$$

Line Charge Density



Line charge, ρ_L (C/m)

$$\vec{E}(\vec{r}) = \int_{\mathcal{L}} \frac{\rho_L(\vec{r}\,')(\vec{r}-\vec{r}\,')\,d\ell'}{4\pi\epsilon_0 \|\vec{r}-\vec{r}\,'\|^3}$$

$$Q = \int_{\mathcal{L}} \rho_L(\vec{r}\,')\,d\ell'$$

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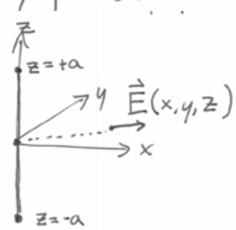
Example Problem: Line Charge

Example 1 Line Charge 5

If charge is uniformly distributed on a line from $z=-a$ to $z=+a$ with density ρ_L (C/m), what is the field on the xy-plane? . . .



without loss of generality
solve for $\vec{r} = x\hat{x} + 0\hat{y} + 0\hat{z}$

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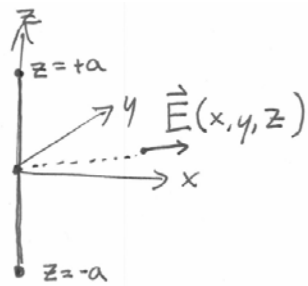

without loss of generality,
solve for $\vec{r} = x\hat{x} + 0\hat{y} + 0\hat{z}$

integrate along z-axis

$dl' = dz'$

$\vec{r}\,' = z'\hat{z}$

$\|\vec{r} - \vec{r}\,'\| = \sqrt{x^2 - z'^2}$

$\rho_L(\vec{r}\,') = \rho_L$

$\vec{r} - \vec{r}\,' = x\hat{x} - z'\hat{z}$

Symmetry and Simplification

due to symmetry, only $\hat{x}$ term will be non-zero.

Final computation

$$\vec{E}(x, \overset{0}{y}, \overset{0}{z}) = \int_{-a}^{+a} \frac{\rho_L (x\hat{x} - \underbrace{z'\hat{z}}_{\to 0})}{4\pi\epsilon_0 [x^2 - z'^2]^{3/2}}\, dz'$$

$$= \frac{\rho_L x \hat{x}}{4\pi\epsilon_0} \int_{-a}^{+a} \frac{dz'}{[x^2 - z'^2]^{3/2}}$$

Solution

Now let's look in the back of the calc book:

$$\int_{-a}^{a} \frac{dz'}{[x^2+z'^2]^{3/2}} = \frac{1}{x^2}\left[\frac{z'}{\sqrt{x^2+z'^2}}\right]_{-a}^{a}$$

$$= \frac{1}{x^2}\frac{2a}{\sqrt{x^2+a^2}}$$

Final Answer:

$$\vec{E}(x,0,0) = \frac{\rho_L x \hat{x}}{4\pi\epsilon_0}\cdot\frac{1}{x^2}\frac{2a}{\sqrt{x^2+a^2}}$$

$$= \frac{\rho_L a \hat{x}}{2\pi\epsilon_0 x\sqrt{x^2+a^2}}$$

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Express in Cylindrical Coordinates

Re-write with $\rho = \sqrt{x^2+y^2}$

$$\vec{E}(x,y,0) = \frac{\overbrace{(\rho_L \cdot 2a)}^{Q}}{4\pi\epsilon_0\,\rho\sqrt{\rho^2+a^2}}\hat{\rho}$$

Q total charge

$\hat{\rho}$ points away from z-axis

$$\vec{E}(x,y,0) = \frac{Q}{4\pi\epsilon_0\rho a}\hat{\rho}$$

Close-in $\lim \rho \to 0$

$$\vec{E}(x,y,0) = \frac{Q}{4\pi\epsilon_0\rho^2}\hat{\rho}$$

like a point charge!

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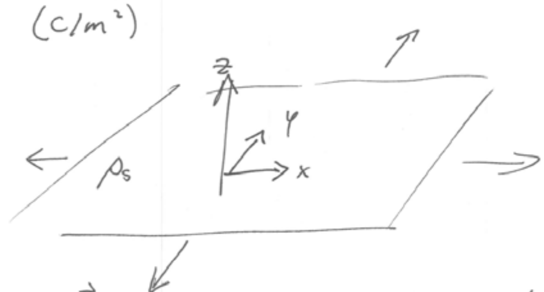
Example: Infinite Sheet Charge

Example 2: Sheet of Charge 7/

A uniform, infinite sheet of surface charge ρ_s (C/m²)



$$\vec{E}(\underbrace{x,y,z}_{\vec{r}}) = \int_S \frac{\rho_s (\vec{r}-\vec{r}\,')\, ds'}{4\pi\epsilon_0 \|\vec{r}-\vec{r}\,'\|^3}$$

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Symmetry and Simplification

integrating in x' and y'

$ds' = dx'\,dy'$ $\qquad \vec{r}\,' = x'\hat{x} + y'\hat{y} + 0\hat{z}$

$\|\vec{r}-\vec{r}\,'\| = \sqrt{z^2 + x'^2 + y'^2}$

due to observation at $(0,0,z)$ without any loss of generality (translational invariance with respect to x and y).

$\rho_s(\vec{r}\,') = \rho_s$

$\vec{r} - \vec{r}\,' = z\hat{z} - \underbrace{(x'\hat{x} + y'\hat{y})}$

note that these components will integrate to 0 due to symmetry

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Final Integral

Final Set-up:

$$\vec{E}(0,0,z) = \hat{z}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy' \frac{\rho_s z}{4\pi\epsilon_0\left[z^2+x'^2+y'^2\right]^{3/2}}$$

$$= \frac{\rho_s z\,\hat{z}}{4\pi\epsilon_0}\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy' \frac{1}{\left[z^2+x'^2+y'^2\right]^{3/2}}$$

Evaluation

first integral (w.r.t. x')

$$\text{Integration} = \int_{-\infty}^{\infty}\frac{dy'}{(z^2+y'^2)}\underbrace{\left[\frac{x'}{\sqrt{z^2+x'^2+y'^2}}\right]_{-\infty}^{\infty}}_{=2}$$

$$= 2\int_{-\infty}^{\infty}\frac{dy'}{(z^2+y'^2)} = \frac{2}{z}\underbrace{\tan^{-1}\frac{y'}{z}\Bigg|_{-\infty}^{\infty}}_{\pi}$$

Uniform Field Above and Below the Sheet

Final Answer

$$\vec{E}(0,0,z) = \frac{\rho_s \hat{z}}{2\epsilon_0} \quad \text{for } z > 0$$

Thus

$$\vec{E}(x,y,z) = \frac{\rho_s \hat{z}}{2\epsilon_0}$$

constant in all of space!

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