<u>Curriculum Topic</u> :
Electrostatic Fields

<u>ESF2</u> : Coulomb's Law in Advanced Coordinate Systems

Module Outline:	
Prerequisite Skills	Competencies
Supplemental Reading and Resources	Assessments
Power Point Slides and Notes	

Prerequisite Skills

Prerequisites / Requirements:
ESF1 Coulomb's Law in Advanced Coordinate Systems

Competencies

Competency ESF.2: **Compute electric field from a description of charge in space using the spherical and cylindrical coordinate systems.**

Competency Builders:
ESF.2.1 Construct field integrals given an arbitrary charge distribution function using integration in the cylindrical coordinate system
ESF.2.2 Construct field integrals given an arbitrary charge distribution function using integration in the spherical coordinate system

Supplemental Reading and Resources

Supplemental Reading Materials:

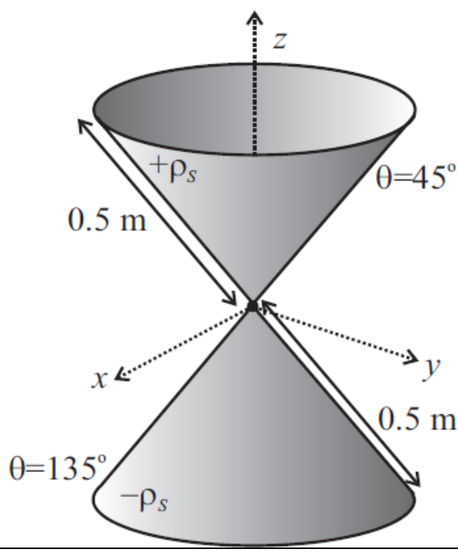
Prof. Andrew Peterson's Lecture Notes (Fields and Waves Lectures 2 and 3)

Assessments

The following questions and exercises may serve as either pre-assessment or post-assessment tests to evaluate student knowledge.

Question: ESF.2.1 *Competency:* ESF.2.1

(4) **Conical Charge Dipole:** A biconical surface of charge is defined by the regions $\theta = 45°$, $\theta = 135°$, and $r < 0.5$m. The upper cone has a positive, uniform surface charge density of $+\rho_S$ and the lower cone has a negative, uniform surface charge density of $-\rho_S$. Derive an expression for calculating the electrostatic field $\vec{E}$ for point on the xy plane due to this charge distribution. *Simplify as much as possible* without evaluating the final integral(s). (**30 points**)



Answer:

(4) **Conical Charge Dipole:**

Lots of partial credit given for this one! You could save yourself a lot of trouble if you recognize that the electric field components in $\hat{x}$ and $\hat{y}$ due to the positive charges on the top cone will perfectly cancel out with the corresponding components due to the negative charges on the bottom cone. If you did not see this symmetry, you could still discover this cancellation with the standard technique for setting these problems up. The full-length solution is given below:

$$\vec{E}(\vec{r}) = \int\limits_S \frac{\overbrace{\rho_S(\vec{r}')}^{\rho_0}(\vec{r}-\vec{r}')\,dS}{4\pi\epsilon||\vec{r}-\vec{r}'||^3}$$

$$\vec{E}(x,y,0) = \frac{\rho_0}{4\pi\epsilon}\int\limits_{\substack{\text{Top}\\\text{Cone}}} \frac{(\overbrace{x\hat{x}+y\hat{y}+0\hat{z}}^{\vec{r}}\,\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{x\hat{x}+y\hat{y}+0\hat{z}}_{\vec{r}}\,\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3} - \frac{\rho_0}{4\pi\epsilon}\int\limits_{\substack{\text{Bot.}\\\text{Cone}}} \frac{(\overbrace{x\hat{x}+y\hat{y}+0\hat{z}}^{\vec{r}}\,\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{x\hat{x}+y\hat{y}+0\hat{z}}_{\vec{r}}\,\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3}$$

$$= \frac{\rho_0}{4\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{[(x-r\sin 45^\circ\cos\phi)\hat{x} + (y-r\sin 45^\circ\sin\phi)\hat{y} - r\cos 45^\circ\hat{z}]}{[(x-r\sin 45^\circ\cos\phi)^2 + (y-r\sin 45^\circ\sin\phi)^2 + r^2\cos^2 45^\circ]^{\frac{3}{2}}} r\sin 45^\circ\,d\phi\,dr$$

$$- \frac{\rho_0}{4\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{[(x-r\sin 135^\circ\cos\phi)\hat{x} + (y-r\sin 135^\circ\sin\phi)\hat{y} - r\cos 135^\circ\hat{z}]}{[(x-r\sin 135^\circ\cos\phi)^2 + (y-r\sin 135^\circ\sin\phi)^2 + r^2\cos^2 135^\circ]^{\frac{3}{2}}} r\sin 135^\circ\,d\phi\,dr$$

$$= \frac{\rho_0}{4\sqrt{2}\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)\hat{x} + (y-r\frac{1}{\sqrt{2}}\sin\phi)\hat{y} - r\frac{1}{\sqrt{2}}\hat{z}\right]}{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)^2 + (y-r\frac{1}{\sqrt{2}}\sin\phi)^2 + \frac{r^2}{2}\right]^{\frac{3}{2}}} r\,d\phi\,dr$$

$$- \frac{\rho_0}{4\sqrt{2}\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)\hat{x} + (y-r\frac{1}{\sqrt{2}}\sin\phi)\hat{y} + r\frac{1}{\sqrt{2}}\hat{z}\right]}{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)^2 + (y-r\frac{1}{\sqrt{2}}\sin\phi)^2 + \frac{r^2}{2}\right]^{\frac{3}{2}}} r\,d\phi\,dr$$

$$= -\frac{\rho_0\hat{z}}{4\pi\epsilon}\int_0^{\frac{1}{2}} r^2\,dr\int_0^{2\pi} d\phi \frac{1}{\left[(x-\frac{r}{\sqrt{2}}\cos\phi)^2 + (y-\frac{r}{\sqrt{2}}\sin\phi)^2 + \frac{r^2}{2}\right]^{\frac{3}{2}}}$$

Question: ESF.2.2 *Competency:* ESF.2.2

Spherical Coordinates: We know that for spherical coordinates, the unit vectors that describe field components point in different directions in space, depending on the observation point. If you know that $\theta = 0$ and $\phi = 0$ for the observation point, what are $\hat{a}_r$, $\hat{a}_\phi$, and $\hat{a}_\theta$ in terms of Cartesian unit vectors? (10 points)

Answer:

$\hat{a}_r = \hat{a}_z$, $\hat{a}_\theta = \hat{a}_x$, $\vec{a}_\phi = \hat{a}_y$

Question: ESF.2.3 *Competency:* ESF.2.2

Fields in the Ionosphere: The ionosphere of earth occurs above 80 km from the surface of the earth. (The earth, itself, has a radius of 6000 km). The ionosphere is populated with free electrons that were dissociated from molecules and atoms by the solar radiation. You find one model in the research literature for the distribution of charge density (C/m^3) as a function of radius and elevation angle:

$$\rho(r,\phi,\theta) = \begin{cases} 0 & r < 6080 \text{ km} \\ -10^{-10}\sin(\theta) & 6080 \text{ km} \leq r < 6180 \text{ km} \\ 0 & r \geq 6180 \text{ km} \end{cases}$$

(a) What is the total amount of charge in the ionosphere?

(b) We are interested in calculating the electric field in deep space due to all charges in the ionosphere on earth. For this we can approximate the earth as a point charge. Write the equation for total electric field due to the ionosopheric charge as a function of $\vec{r}$, assuming that the earth is at the origin of the coordinate system.

(c) Set up the integral (but do not evaluate) for calculating the exact electric field for the scenario in (b), this time *not* making the point charge approximation.

Answer:

(3) **Fields in Ionosphere:** The total charge in the ionosphere can be calculated using the following integral:

$$Q = \int_{6080}^{6180}\int_{0}^{2\pi}\int_{0}^{\pi} \left[-10^{-10}\sin(\theta)\right] \underbrace{r^2 \sin(\theta)\, dr\, d\phi\, d\theta}_{dV} = -2\pi \times 10^{-10} \int_{6080}^{6180}\int_{0}^{\pi} r^2 \sin^2(\theta)\, dr\, d\theta$$

This can be simplified more, but on a 50-minute test there is no need to take it any further. For the field in outer space due to this charge in (b), we (approximately) use the Coulomb point charge equation:

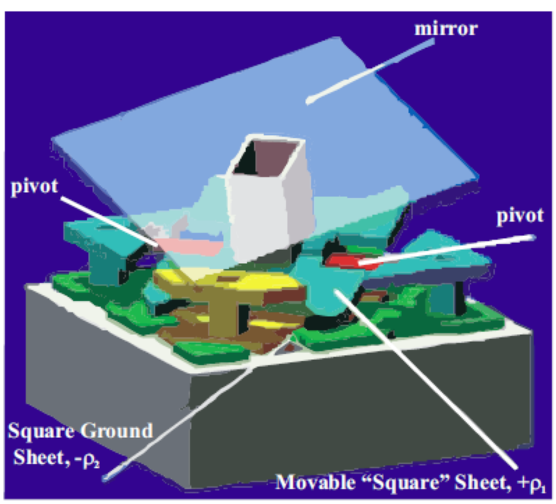
$$\vec{E}(\vec{r}) = \frac{Q\vec{r}}{4\pi\epsilon_0|\vec{r}|^3} \quad \text{or} \quad \vec{E}(\vec{r}) = \frac{Q\hat{a}_r}{4\pi\epsilon_0|\vec{r}|^2}$$

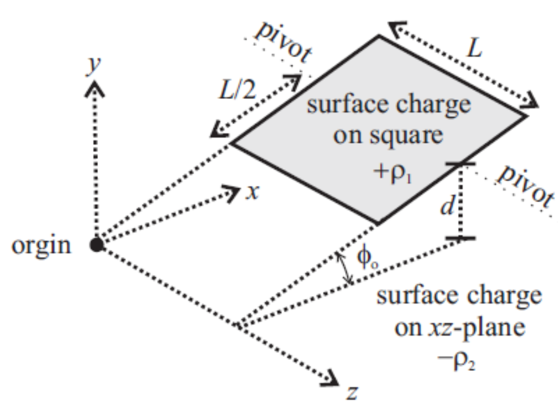
If we wanted to calculate this field exactly, we would use the following integral:

$$\vec{E}(\vec{r}) = \int_{6080}^{6180}\int_{0}^{2\pi}\int_{0}^{\pi} \frac{\left[-10^{-10}\sin(\theta')\right](\vec{r}-\vec{r}')}{4\pi\epsilon_0|\vec{r}-\vec{r}'|^3} \underbrace{r'^2 \sin(\theta')\, dr'\, d\phi'\, d\theta'}_{dV}$$

Question: ESF.2.4 *Competency:* ESF.2.1

DLP Chip: Below is a 3D drawing and schematic of a dynamic light projection (DLP) mirror unit. The mirror is mounted on a pivoting metallic *square* (for modeling purposes) of side length L with uniform surface charge density $+\rho_1$. This square may tilt at an arbitrary angle ϕ_o with respect to an infinite (for modeling purposes) ground plane with uniform surface charge density $-\rho_2$. A value of $\phi_o = 0$ corresponds to the square and mirror perfectly parallel to the ground plane. The pivots bisect the square sheet, with the pivot points resting above the infinite ground plane by a distance d. Set-up any combination of integrals or expressions that solve for *total* electric field at an arbitrary point (x, y, z) in free space around the device. Simplify as much as possible, but do not evaluate non-trivial integral(s). **(30 points)**





Answer:

The total electric field is the superposition of the infinite plane charge (a result we know from class and on the formula sheet) and the tilted square:

$$\vec{E}(x,y,z) = \frac{-\rho_2}{2\epsilon_o}\hat{y} + \iint_S \frac{\rho_S(\vec{r}')(\vec{r}-\vec{r}')dS}{4\pi\epsilon\|\vec{r}-\vec{r}'\|^3}$$

$$= \frac{-\rho_2}{2\epsilon_o}\hat{y} + \frac{\rho_1}{4\pi\epsilon_o}\int_0^L dz' \int_{\frac{d}{\sin\phi_o}-\frac{L}{2}}^{\frac{d}{\sin\phi_o}+\frac{L}{2}} d\rho' \frac{[(x-\rho'\cos\phi_o)\hat{x} + (y-\rho'\sin\phi_o)\hat{y} + (z-z')\hat{z}]}{[(x-\rho'\cos\phi_o)^2 + (y-\rho'\sin\phi_o)^2 + (z-z')^2]^{\frac{3}{2}}}$$

which is valid for $y > 0$.

Question: ESF.2.5 *Competency:* ESF.2.1,2

Coordinate Systems: Answer the following questions about non-Cartesian 3D coordinate systems (**15 points**).

(a) Write expressions for the cylindrical coordinate unit vectors $\hat{\rho}$, $\hat{\phi}$, $\hat{z}$ in terms of Cartesian unit vectors for an observation point at $\rho = 6$, $\phi = 90°$, and $z = 7$.

(b) Write expressions for the spherical coordinate unit vectors $\hat{r}$, $\hat{\phi}$, $\hat{\theta}$ in terms of Cartesian unit vectors for an observation point at $r = 2$, $\phi = 180°$, and $\theta = 90°$.

(c) Write expressions for the spherical coordinate unit vectors $\hat{r}$, $\hat{\phi}$, $\hat{\theta}$ in terms of Cartesian unit vectors for an observation point at $r = 12$, $\phi = 0°$, and $\theta = 0°$.

Answer:

(a)

$$\hat{\rho} = \hat{y} \qquad \hat{\phi} = -\hat{x} \qquad \hat{z} = \hat{z}$$

(b)

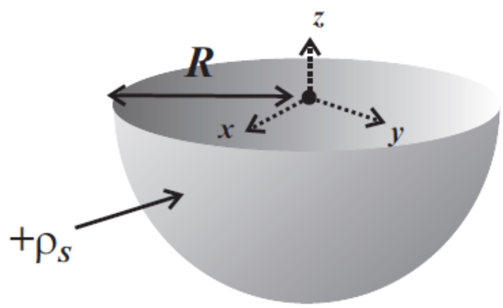
$$\hat{r} = -\hat{x} \qquad \hat{\phi} = -\hat{y} \qquad \hat{\theta} = -\hat{z}$$

(c)

$$\hat{r} = \hat{z} \qquad \hat{\phi} = \hat{y} \qquad \hat{\theta} = \hat{x}$$

Question: ESF.2.6 *Competency:* ESF.2.2

Electrostatic Integrals: Below is a bowl of uniform *surface* charge density, ρ_s, defined by the half-surface of a sphere of radius R, centered at the origin, that lies below the xy-plane. Answer all questions based on this scenario. (**35 points**).



(a) Calculate the total amount of charge on the bowl surface in terms of ρ_s and R. Fully evaluate this answer. (**10 points**)

(b) Write an expression for the electric field observed at the origin, $\vec{E}(0,0,0)$, due to the presence of the charged bowl. You must simplify as much as possible. Note: this answer can and should be solved completely using basic integration operations. (**20 points**)

(c) One way to check your answer in the previous step is to test limiting cases against other canonical problems with known analytic solutions. In class, we derived the result for an infinite plane. Under what geometrical conditions does the answer in (b) correspond to this problem? (**5 points**)

Answer:

(a) Integrate over the spherical surface charge to find the total charge:

$$Q = \int_{\pi/2}^{\pi} R\, d\theta' \int_{0}^{2\pi} R\, \sin\theta'\, d\phi\, \rho_s = 2\pi R^2 \rho_s$$

Of course, this is just half the area of a sphere of radius R multiplied by the uniform surface charge density ρ_s; full credit was given if you simply solved this geometrically.

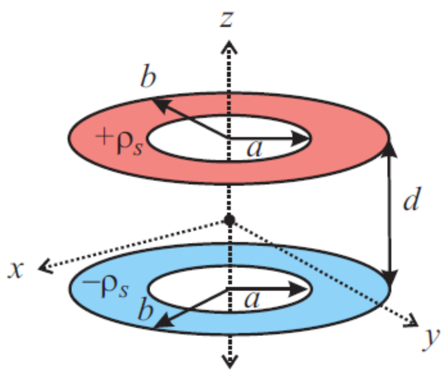
(b) The following answer received full credit

$$\begin{aligned}
\vec{E}(0,0,0) &= \int_{\pi/2}^{\pi} R\, d\theta' \int_{0}^{2\pi} R\, \sin\theta'\, d\phi\, \frac{\rho_s}{4\pi\epsilon} \frac{(\vec{r}-\vec{r}')}{\underbrace{\|\vec{r}-\vec{r}'\|^3}_{=\, R \text{ for all } \vec{r}'}} \\
&= \frac{\rho_s}{4\pi\epsilon R} \int_{\pi/2}^{\pi} d\theta' \int_{0}^{2\pi} \sin\theta'\, d\phi \left(\underbrace{-x'\hat{x} - y'\hat{y}}_{=\, 0 \text{ by symmetry}} \; -z'\hat{z} \right) \\
&= \frac{\rho_s}{4\pi\epsilon R} \int_{\pi/2}^{\pi} d\theta' \int_{0}^{2\pi} \sin\theta'\, d\phi\, (-R\cos\theta'\, \hat{z}) \\
&= -\frac{\rho_s}{2\epsilon R}\hat{z} \int_{\pi/2}^{\pi} d\theta' \sin\theta'\; (R\, \cos\theta') \\
&= -\frac{\rho_s}{2\epsilon}\hat{z} \int_{\pi/2}^{\pi} d\theta' \sin\theta'\; \cos\theta' \\
&= -\frac{\rho_s}{4\epsilon}\hat{z}\, \sin^2\theta' \Big|_{\pi/2}^{\pi} \\
&= \frac{\rho_s}{4\epsilon}\hat{z}
\end{aligned}$$

(c) Mostly resembles our constant electric field solution for the plane of charge when the sphere is infinite $R \to \infty$.

Question: ESF.2.7 *Competency:* ESF.2.1

(5) **Charge Disks:** A pair of flat rings have uniform surface charge density with opposite polarities; a density of $+\rho_S$ is distributed on the top ring while a density of $-\rho_S$ is distributed on the bottom ring. The rings are separated by a distance d and have inner and outer radii of a and b, respectively. Given that the whole assembly is centered on the origin, find an expression for electrostatic field along the z-axis – i.e. find $\vec{E}(0,0,z)$. *Simplify as much as possible* without evaluating the final integral(s). (**30 points**)



Answer:

$$\vec{E}(\vec{r}) = \int_S \frac{\overbrace{\rho_S(\vec{r}')}^{\rho_S}(\vec{r}-\vec{r}')\,dS}{4\pi\epsilon||\vec{r}-\vec{r}'||^3}$$

$$\vec{E}(0,0,z) = \frac{\rho_S}{4\pi\epsilon}\int_{\substack{\text{Top}\\\text{Disk}}} \frac{(\overbrace{0\hat{x}+0\hat{y}+z\hat{z}}^{\vec{r}}\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{0\hat{x}+0\hat{y}+z\hat{z}}_{\vec{r}}\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3} - \frac{\rho_S}{4\pi\epsilon}\int_{\substack{\text{Bot.}\\\text{Disk}}} \frac{(\overbrace{0\hat{x}+0\hat{y}+z\hat{z}}^{\vec{r}}\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{0\hat{x}+0\hat{y}+z\hat{z}}_{\vec{r}}\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3}$$

$$= \frac{\rho_S}{4\pi\epsilon}\int_a^b\int_0^{2\pi} \frac{\overbrace{(-x')\hat{x}+(-y')\hat{y}}^{\text{symmetry}\to 0}+(z-\frac{D}{2})\hat{z}}{\left[\underbrace{(x')^2+(y')^2}_{\rho'^2}+(z-\frac{D}{2})^2\right]^{\frac{3}{2}}}\rho'\,d\phi'\,d\rho' - \frac{\rho_S}{4\pi\epsilon}\int_a^b\int_0^{2\pi} \frac{\overbrace{(-x')\hat{x}+(-y')\hat{y}}^{\text{symmetry}\to 0}+(z+\frac{D}{2})\hat{z}}{\left[\underbrace{(x')^2+(y')^2}_{\rho'^2}+(z+\frac{D}{2})^2\right]^{\frac{3}{2}}}\rho'\,d\phi'\,d\rho'$$

$$= \frac{\rho_S\hat{z}}{4\pi\epsilon}\left(\int_a^b\int_0^{2\pi} \frac{(z-\frac{D}{2})\rho'\,d\phi'\,d\rho'}{[\rho'^2+(z-\frac{D}{2})^2]^{\frac{3}{2}}} - \int_a^b\int_0^{2\pi} \frac{(z+\frac{D}{2})\rho'\,d\phi'\,d\rho'}{[\rho'^2+(z+\frac{D}{2})^2]^{\frac{3}{2}}}\right)$$

Full credit for this problem was given for reaching this step. The intrepid could continue on for the bonus +5:

$$\vec{E}(0,0,z) = \frac{\rho_S\hat{z}}{2\epsilon}\left(\int_a^b \frac{(z-\frac{D}{2})\rho'\,d\rho'}{[\rho'^2+(z-\frac{D}{2})^2]^{\frac{3}{2}}} - \int_a^b \frac{(z+\frac{D}{2})\rho'\,d\rho'}{[\rho'^2+(z+\frac{D}{2})^2]^{\frac{3}{2}}}\right)$$

$$= \frac{\rho_S\hat{z}}{2\epsilon}\left(\frac{(z+\frac{d}{2})}{\sqrt{b^2+(z+\frac{d}{2})^2}} - \frac{(z+\frac{d}{2})}{\sqrt{a^2+(z+\frac{d}{2})^2}} - \frac{(z-\frac{d}{2})}{\sqrt{b^2+(z-\frac{d}{2})^2}} + \frac{(z-\frac{d}{2})}{\sqrt{a^2+(z-\frac{d}{2})^2}}\right)$$