ESF2: Coulomb's Law in Advanced Coordinate Systems

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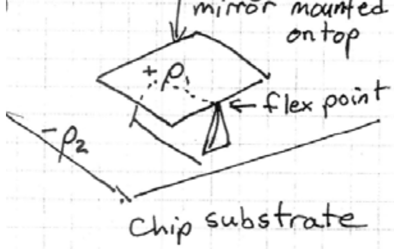
Example: Forces on a DLP Mirror

Dynamic Light Projection (DLP) Mirror



$+\rho_1$ and $-\rho_2$ are Surface charges

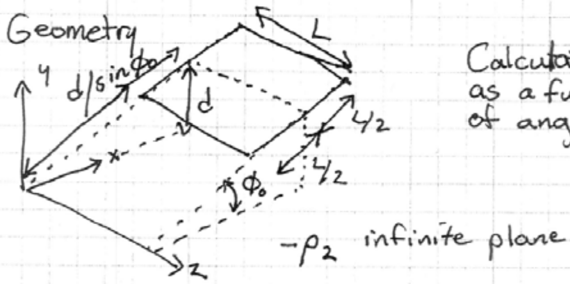
MEMs device

Micro-

$\rho_1 = 0 \rightarrow$ flat table, mirror in place for reflection

$\rho_2 > 0 \rightarrow$ attractive force bends down

Superposition



Calculate fields as a function of angle

$$\vec{E}(x,y,z) = \underbrace{\vec{E}_2}_{\text{due to infinite plane}} + \underbrace{\vec{E}_1}_{\text{due to charge table}}$$

$$\vec{E}_2 = \frac{-\rho_2}{2\epsilon_0}\,\hat{y}$$ due to infinite charge sheet

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Basics of Cylindrical Coordinates

Point of observation

<u>Cylindrical</u>	<u>Cartesian</u>
$\rho = \sqrt{x^2+y^2}$	$x = \rho\cos\phi$
$\phi = \begin{cases} \tan^{-1}\frac{y}{x} & x \ge 0 \\ \pi + \tan^{-1}\frac{y}{x} & x < 0 \end{cases}$	$y = \rho\sin\phi$
$z = z$	$z = z$

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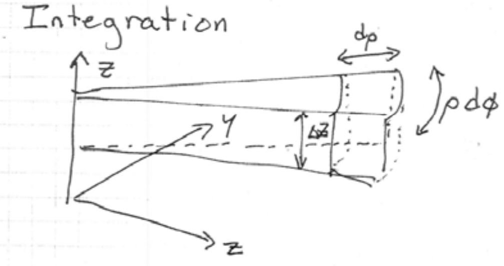
Integration and Vector Expression

Integration



Unit Vectors

$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$

$\hat{z} = \hat{z}$

Basic Recipe for Solving Field Problems

1) Identify Integral Type, Argument, Kernal (Greens Function)
2) Define region of integration $\vec{r}\,' \in R$
3) Write integral in terms of $\vec{r}, \vec{r}\,'$
4) Insert Scalar Position variables
5) Simplify

$$\vec{E}_1(x,y,z) = \int_{Surf} \frac{\rho_1 \, (\vec{r} - \vec{r}\,') \, dS}{4\pi\epsilon_0 \|\vec{r} - \vec{r}\,'\|^3}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

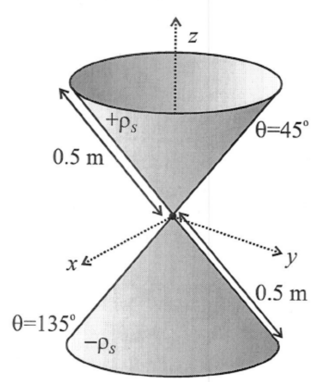
$$= \rho'\cos\phi'\,\hat{x} + \rho'\sin\phi'\,\hat{y} + z'\hat{z}$$

note $\phi' = \phi_0$ constant

$$\vec{E}_1 = \int_0^L dz' \int_{\frac{d}{\sin\phi_0} - L/2}^{\frac{d}{\sin\phi_0} + L/2} d\rho' \; \frac{\rho_1\left[(x-\rho'\cos\phi_0)\hat{x} + (y-\rho'\sin\phi_0)\hat{y} + (z-z')\hat{z}\right]}{4\pi\epsilon_0\left[(x-\rho'\cos\phi_0)^2 + (y-\rho'\sin\phi_0)^2 + (z-z')^2\right]^{3/2}}$$

Example 2: Conical Charge

(4) **Conical Charge Dipole:** A biconical surface of charge is defined by the regions $\theta = 45°$, $\theta = 135°$, and $r < 0.5$m. The upper cone has a positive, uniform surface charge density of $+\rho_S$ and the lower cone has a negative, uniform surface charge density of $-\rho_S$. Derive an expression for calculating the electrostatic field $\vec{E}$ for point on the xy plane due to this charge distribution. *Simplify as much as possible* without evaluating the final integral(s). (**30 points**)



Basics of Spherical Coordinates

$(x, y, z) \longrightarrow (r, \phi, \theta)$

r - distance from origin

ϕ - azimuth angle

θ - elevation angle (measured from z-axis)

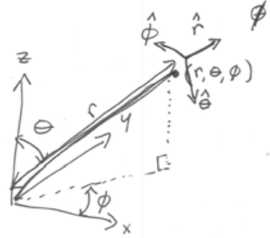
Note: emag convention is opposite math

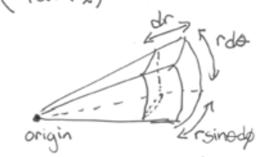
$x = r \sin\theta \cos\phi$ $\qquad r = \sqrt{x^2+y^2+z^2}$

$y = r \sin\theta \sin\phi$ $\qquad \theta = \mathrm{acos}\left(\frac{z}{r}\right)$ or $\tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right)$

$z = r\cos\theta$

$$\phi = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \geq 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x \leq 0 \end{cases}$$





integration elements

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Vector Expression in Spherical Coordinates

Vectors in Spherical Coordinates

$\vec{E}(\vec{r}) = E_r(\vec{r})\hat{r} + E_\phi(\vec{r})\hat{\phi} + E_\theta(\vec{r})\hat{\theta}$

How to express in cartesian coordinates?

	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{x}$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{y}$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{z}$	$\cos\theta$	$-\sin\theta$	0

dot products

$\vec{E}(\vec{r}) = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$

$E_x = \hat{x}\cdot\vec{E} = E_r(\hat{x}\cdot\hat{r}) + E_\phi(\hat{x}\cdot\hat{\phi}) + E_\theta(\hat{x}\cdot\hat{\theta})$

$= E_r\sin\theta\cos\phi - E_\phi\sin\phi + E_\theta\cos\theta\cos\phi$

and similarly for E_y and E_z

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Evaluate with Superposition

$$\vec{E}(\vec{r}) = \int_S \frac{\overbrace{\rho_S(\vec{r}')}^{\rho_0}(\vec{r}-\vec{r}')\,dS}{4\pi\epsilon||\vec{r}-\vec{r}'||^3}$$

$$\vec{E}(x,y,0) = \frac{\rho_0}{4\pi\epsilon}\int_{\substack{\text{Top}\\\text{Cone}}} \frac{(\overbrace{x\hat{x}+y\hat{y}+0\hat{z}}^{\vec{r}}\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{x\hat{x}+y\hat{y}+0\hat{z}}_{\vec{r}}\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3} - \frac{\rho_0}{4\pi\epsilon}\int_{\substack{\text{Bot.}\\\text{Cone}}} \frac{(\overbrace{x\hat{x}+y\hat{y}+0\hat{z}}^{\vec{r}}\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{x\hat{x}+y\hat{y}+0\hat{z}}_{\vec{r}}\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3}$$

$$= \frac{\rho_0}{4\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{[(x-r\sin 45^\circ\cos\phi)\hat{x}+(y-r\sin 45^\circ\sin\phi)\hat{y}-r\cos 45^\circ\hat{z}]}{[(x-r\sin 45^\circ\cos\phi)^2+(y-r\sin 45^\circ\sin\phi)^2+r^2\cos^2 45^\circ]^{\frac{3}{2}}}r\sin 45^\circ\,d\phi\,dr$$

$$- \frac{\rho_0}{4\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{[(x-r\sin 135^\circ\cos\phi)\hat{x}+(y-r\sin 135^\circ\sin\phi)\hat{y}-r\cos 135^\circ\hat{z}]}{[(x-r\sin 135^\circ\cos\phi)^2+(y-r\sin 135^\circ\sin\phi)^2+r^2\cos^2 135^\circ]^{\frac{3}{2}}}r\sin 135^\circ\,d\phi\,dr$$

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Final Answer

$$= \frac{\rho_0}{4\sqrt{2}\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)\hat{x}+(y-r\frac{1}{\sqrt{2}}\sin\phi)\hat{y}-r\frac{1}{\sqrt{2}}\hat{z}\right]}{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)^2+(y-r\frac{1}{\sqrt{2}}\sin\phi)^2+\frac{r^2}{2}\right]^{\frac{3}{2}}}r\,d\phi\,dr$$

$$- \frac{\rho_0}{4\sqrt{2}\pi\epsilon}\int_0^{\frac{1}{2}}\int_0^{2\pi} \frac{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)\hat{x}+(y-r\frac{1}{\sqrt{2}}\sin\phi)\hat{y}+r\frac{1}{\sqrt{2}}\hat{z}\right]}{\left[(x-r\frac{1}{\sqrt{2}}\cos\phi)^2+(y-r\frac{1}{\sqrt{2}}\sin\phi)^2+\frac{r^2}{2}\right]^{\frac{3}{2}}}r\,d\phi\,dr$$

$$= -\frac{\rho_0\hat{z}}{4\pi\epsilon}\int_0^{\frac{1}{2}}r^2\,dr\int_0^{2\pi}d\phi\frac{1}{\left[(x-\frac{r}{\sqrt{2}}\cos\phi)^2+(y-\frac{r}{\sqrt{2}}\sin\phi)^2+\frac{r^2}{2}\right]^{\frac{3}{2}}}$$

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