ESF4: Gauss's Law in Differential Form

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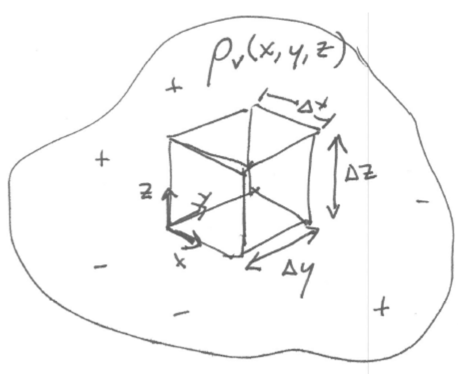
Apply Gauss's Law Integral Form to Small Prism



Assume vanishingly small volume for Gauss's Law integral

Electric Flux Integral

$$\oint_S \vec{D}\cdot d\hat{n} = \text{Flux through six sides of a rectangular prism.}$$

$$\begin{aligned}\oint_S \vec{D}\cdot d\hat{n} = {} & D_x(x+\Delta x, y, z)\,\Delta z\,\Delta y - D_x(x, y, z)\,\Delta z\,\Delta y \\ & + D_y(x, y+\Delta y, z)\,\Delta x\,\Delta z - D_y(x, y, z)\,\Delta x\,\Delta z \\ & + D_z(x, y, z+\Delta z)\,\Delta x\,\Delta y - D_z(x, y, z)\,\Delta x\,\Delta y\end{aligned}$$

$$\text{Enclosed Charge} = \rho_v(x, y, z)\,\Delta x\,\Delta y\,\Delta z$$

Differential Form of Gauss's Law

Thus

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \frac{D_x(x+\Delta x, y, z) - D_x(x, y, z)}{\Delta x} + \frac{D_y(x, y+\Delta y, z) - D_y(x, y, z)}{\Delta y} + \frac{D_z(x, y, z+\Delta z) - D_z(x, y, z)}{\Delta z} = \rho_v(x, y, z)$$

$$\rho_v(x, y, z) = \frac{\partial D_x(x, y, z)}{\partial x} + \frac{\partial D_y(x, y, z)}{\partial y} + \frac{\partial D_z(x, y, z)}{\partial z}$$

$$= \nabla\cdot\vec{D}(x, y, z)$$

Divergence Measures Sources

Recall $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

Del operator Shorthand notation

$\nabla \cdot \vec{D} = \rho_v$ – Gauss's Law in Differential or "Point" Form

– One of Maxwell's equations

Divergence $\equiv$ "Sourceiness"

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The Divergence Theorem

Basic Relationship in Vector Calculus

$$\oint_S \vec{D} \cdot d\hat{n}' = \int_V \rho_v \, dv' = \int_V \nabla \cdot \vec{D} \, dv'$$

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Divergence in Non-Cartesian Coordinates

Keep in mind, $\nabla \cdot \vec{D}$ defined differently in spherical and cylindrical coords:

cyl: $$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

sph: $$\nabla \cdot \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$