<u>Curriculum Topic</u> :
Electrostatic Fields

<u>ESF5</u> : Scalar Electric Potential

Module Outline:	
Prerequisite Skills	Competencies
Supplemental Reading and Resources	Assessments
Power Point Slides and Notes	

Prerequisite Skills

Prerequisites / Requirements:
ESF4 Gauss's Law in Differential Form

Competencies

Competency ESF.5: **Calculate the scalar electric potential for charge distributions.**

Competency Builders:
ESF.5.1 Calculate a path integral to compute voltage
ESF.5.2 Calculate electric field from a potential function
ESF.5.3 Calculate electrostatic potential function given an arbitrary charge distribution

Supplemental Reading and Resources

Supplemental Reading Materials:

Prof. Andrew Peterson's Lecture Notes (Fields and Waves Lecture 8)

Assessments

The following questions and exercises may serve as either pre-assessment or post-assessment tests to evaluate student knowledge.

Question: ESF.5.1 *Competency:* ESF.5.1

Two parallel metal plates separated by 0.5 mm have surface charge such that the electric field between them has a uniform magnitude of 2000 V/m in a direction perpendicular to the plates. What is the voltage drop between the plates?

Answer:

10 V

Question: ESF.5.2 *Competency:* ESF.5.2

Electrostatic Charge Distributions: All of the field distributions in this problem are free-space and may be written in the following form:

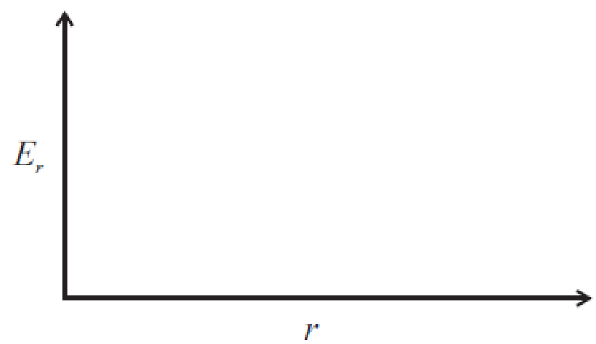
$$\vec{E}(r,\phi,\theta) = E_r(r)\hat{r}$$

Make a rough sketch in the graph provided of $E_r(r)$ for the following charge distributions.

There is a voltage distribution in space of the form:

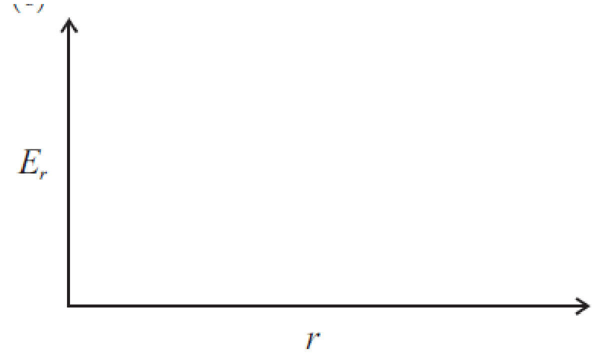
$$V(r,\phi,\theta) = V_o \cos(2\pi r/\lambda)$$

(8 points)

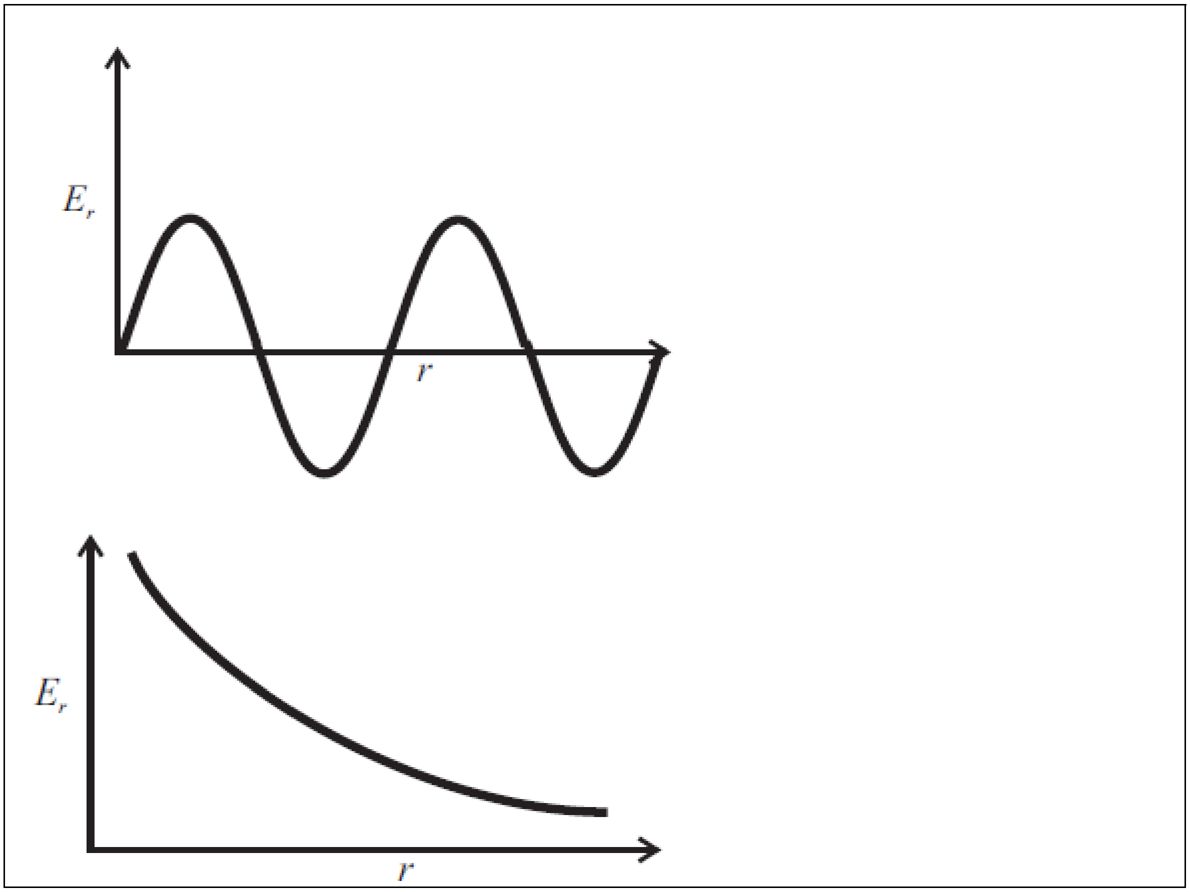


There is a voltage distribution in space of the form:

$$V(r,\phi,\theta) = V_o \exp(-r)$$



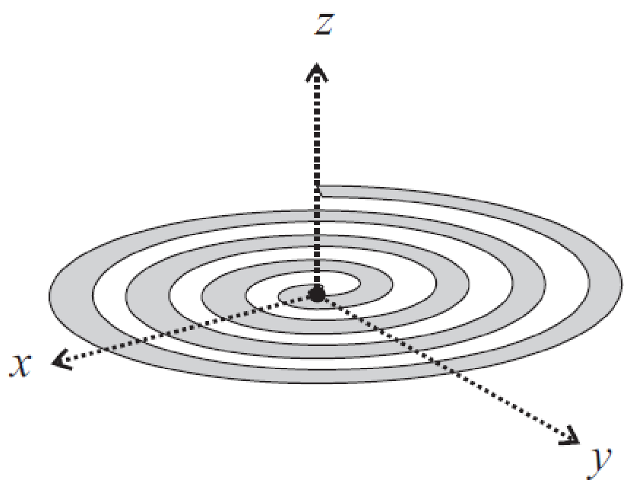
Answer:



E_r
r
E_r
r

Question: ESF.5.3 *Competency:* ESF.5.3

Voltage of a Spiral Charge: Below is a flat, infinite Archimedean spiral of surface charge density. The surface charge density is given by $\rho_s(\rho, \phi) = \rho_o \exp(-\rho)$ (C/m^2) such that the charge density on the spiral arm exponentially decays away from the origin. The region of charge is on the xy-plane, bound by the region $\rho \leq \phi \leq \rho + \phi_o$, where ϕ_o is a constant related to the width of the spiral arm ($\phi_o = 0$ is infinitely thin, $\phi_o = 2\pi$ becomes a solid disk). Calculate the voltage at the origin relative to a zero-reference at infinity. Note: this problem can and should produce a tractable analytic solution (**25 points**).

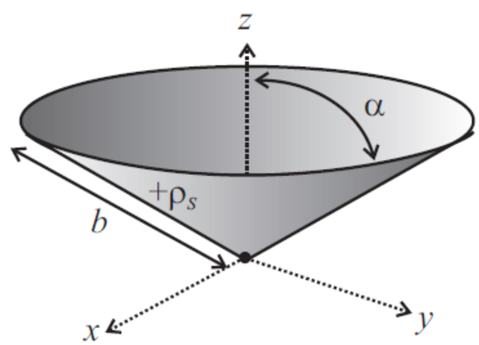


Answer:

$$
\begin{aligned}
V(\vec{r}) &= \iint_S \frac{\rho_S(\vec{r}')dS}{4\pi\epsilon\|\vec{r} - \vec{r}'\|} \\
&= \iint_S \frac{\rho_o \exp(-\rho')\, dS}{4\pi\epsilon\|(0 - x')\hat{x} + (0 - y')\hat{y} + (0 - 0)\hat{z}\|} \\
&= \frac{\rho_o}{4\pi\epsilon} \int_0^{\infty} \int_{\rho'}^{\rho' + \phi_o} \frac{\exp(-\rho')\, \rho'\, d\rho'\, d\phi'}{\rho'} \\
&= \frac{\rho_o \phi_o}{4\pi\epsilon} \int_0^{\infty} \exp(-\rho')\, d\rho' \\
&= \frac{\rho_o \phi_o}{4\pi\epsilon}
\end{aligned}
$$

Question: ESF.5.3 *Competency:* ESF.5.3

Electrostatic Integrals: Below is a cone of uniform surface charge density, ρ_s, with tip at the origin extending out to a distance b, at an angle α to the z-axis. Answer all questions based on this scenario. (**50 points**).



(a) Calculate the total amount of charge on the cone, expressed in variables. Fully evaluate this answer. (**10 points**)

(b) Write an expression for the electric field observed along the z-axis, $\vec{E}(0,0,z)$, due to the presence of the charge cone. You must simplify as much as possible, but you do not need to evaluate the integrals. (**15 points**)

(c) Write an expression for the voltage on the xy-plane, $V(x,y,0)$, due to the presence of the charge cone. You must simplify as much as possible, but you do not need to evaluate the integrals. (**15 points**)

(d) Your results in (b) and (c) are setup for efficient evaluation by a computer. One way to check numerical electromagnetic computations is to test limiting cases of the computation against canonical problems with known analytic solutions. In class, we derived the result for an infinite plane. Under what geometrical conditions of b and α would you be able to check your answer in part (b) against the infinite plane problem? (**5 points**)

(e) Along a similar vein, how might you use the result in (b) to test the results against the electric field of the canonical line charge problem we derived in class (uniform charge on the z-axis from $-a \leq z \leq a$)? (**5 points**)

Answer:

(a) Integrate over the conical surface charge to find the total charge:

$$Q = \int_0^b dr' \int_0^{2\pi} r' \sin\theta|_{\theta=\alpha} \, d\phi' \, \rho_s = \pi \sin\alpha \, b^2 \rho_s$$

(b) The following answer received full credit

$$\vec{E}(0,0,z) = \frac{\rho_s \sin\alpha \, \hat{z}}{4\pi\epsilon} \int_0^b \int_0^{2\pi} \frac{(z - r'\cos\alpha)\, r' dr' \, d\phi'}{\left[r'^2 \sin^2\alpha \cos^2\phi' + r'^2 \sin^2\alpha \sin^2\phi' + (z - r'\cos\alpha)^2\right]^{3/2}}$$

although some astute students were able to further simplify:

$$\vec{E}(0,0,z) = \frac{\rho_s \sin\alpha \, \hat{z}}{2\epsilon} \int_0^b \frac{(z - r'\cos\alpha)\, r' dr'}{\left[r'^2 - 2zr'\cos\alpha + z^2\right]^{3/2}}$$

(c) The reduced, pre-evaluation solution is

$$V(x,y,0) = \frac{\rho_s \sin\alpha \, \hat{z}}{4\pi\epsilon} \int_0^b \int_0^{2\pi} \frac{r' dr' \, d\phi'}{\sqrt{(x - r'\sin\alpha\cos\phi')^2 + (y - r'\sin\alpha\sin\phi') + r'^2\cos^2\alpha}}$$

A seventh-level coordinate system master could have reduced this solution to the following using cylindrical coordinates, which takes advantage of symmetry in the problem:

$$V(\rho,\phi,0) = \frac{\rho_s \sin\alpha \, \hat{z}}{4\pi\epsilon} \int_0^b \int_0^{2\pi} \frac{r' dr' \, d\phi'}{\sqrt{\rho^2 - 2r'\rho\sin\alpha\sin\phi' + r'^2}}$$

(d) $b \to \infty$, $\alpha \to 90^\circ$

(e) Take the solution for $b = a$ and $\alpha = 0^\circ$ and add it to the solution for $b = a$ and $\alpha = 180^\circ$. Another way would be to take the solution for $b = 2a$ and shift the answer down a units along the z-axis.