Curriculum Topic : Electrostatic Fields

ESF6 : Laplace's Equation

Module Outline:	
Prerequisite Skills	Competencies
Supplemental Reading and Resources	Assessments
Power Point Slides and Notes	

Prerequisite Skills

Prerequisites / Requirements:
ESF5 Electrostatic Potential

Competencies

Competency ESF.6: **Apply Laplace's equation to boundary value problems involving electrostatic potential.**

Competency Builders:
ESF.6.1 Derive Poisson's and Laplace's equations
ESF.6.2 Setup boundary value problems for Laplace's equations
ESF.6.3 Apply discrete form of Laplace's equation in a relaxation scheme

Supplemental Reading and Resources

Supplemental Reading Materials:

Prof. Andrew Peterson's Lecture Notes (Fields and Waves Lecture 9)

Assessments

The following questions and exercises may serve as either pre-assessment or post-assessment tests to evaluate student knowledge.

Question: ESF.6.1 *Competency:* ESF.6.1

When does an engineer apply Poisson's equations rather than Laplace's equations?

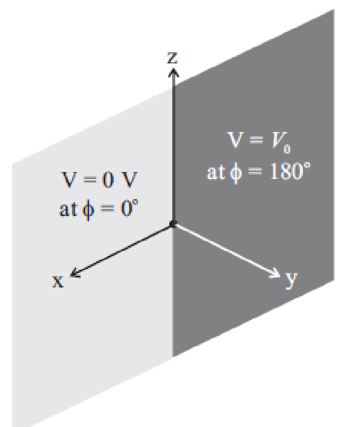
Answer:

When space charges are present in the region, we must use Poisson's equation.

Question: ESF.6.2 *Competency:* ESF.6.2

Laplace's Equation: Two semi-infinite conductive plates are joined at the z-axis and held at 0 and V_0 Volts as shown in the diagram below. Answer the following questions based on this scenario.

(a) Use Laplace's equation to derive the result $V(\rho, \phi, z) = \frac{V_0|\phi|}{\pi}$ for the region $-\pi \leq \phi \leq \pi$. (15 points)



(b) Using the voltage in (a), show that the electric field distribution in space is the following: (10 points)

$$\vec{E}(\rho, \phi, z) = \begin{cases} -\frac{V_0}{\rho\pi}\hat{\phi} & 0 < \phi < \pi \\ \frac{V_0}{\rho\pi}\hat{\phi} & -\pi < \phi < 0 \end{cases} \quad \textbf{or} \quad \vec{E}(\rho, \phi, z) = \frac{2V_0}{\rho\pi}\left[\frac{1}{2} - \mathrm{u}(\phi)\right]\hat{\phi}$$

(c) What is the surface charge density on the plates? (10 points) Hint: $\rho_v(\vec{r}) = \rho_s(\rho, z)\delta(\phi)$

Answer:

(a) We know from symmetry that the voltage will not depend on z or ρ. Thus, applying Laplace's equations:

$$\nabla^2 V(\rho,\phi,z) = \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

First, multiply through by ρ^2 to get the simplified partial-differential equation:

$$\frac{\partial^2 V}{\partial \phi^2} = 0$$

Integrate this twice to produce the general solution:

$$V(\phi) = C_1 + C_2\phi$$

Enforce boundary conditions to solve for the constants C_1 and C_2:

$$V(0) = C_1 = 0V \qquad V(\phi) = C_2\pi = V_0$$

Thus the final solution may be written as

$$V(\rho,\phi,z) = \frac{V_0|\phi|}{\pi}$$

Note that, due to the symmetry of the problem, $V(\phi) = V(-\phi)$; thus, the solution we sketched out in $0 \le \phi \le \pi$ is valid on the other side of the xz-plane as well.

(b) Recognize that $\vec{E} = -\nabla V(\phi)$. The only difficulty in the problem is to make sure we use the gradient formula for the *cylindrical* coordinate system. For $0 \le \phi \le \pi$:

$$\vec{E}(\rho) = -\frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi}\hat{\phi} = -\frac{V_0}{\rho\pi}\hat{\phi}$$

Again, since the solution reflects about the xz-axis, we may write:

$$\vec{E}(\rho,\phi,z) = \begin{cases} -\frac{V_0}{\rho\pi}\hat{\phi} & 0 < \phi < \pi \\ \frac{V_0}{\rho\pi}\hat{\phi} & -\pi < \phi < 0 \end{cases} \quad \text{or} \quad \vec{E}(\rho,\phi,z) = \frac{2V_0}{\rho\pi}\left[\frac{1}{2} - \mathrm{u}(\phi)\right]\hat{\phi}$$

(c) To find charge from an electric field distribution, we apply Gauss's law in differential form: $\nabla \cdot \vec{D} = \rho_v$. Note that the divergence formula in cylindrical coordinates predicts that

$$\nabla \cdot (\epsilon\vec{E}) = \epsilon\frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} = 0$$

for $0 < \phi < \pi$. Does this make sense? Yes, because there is no charge in the space (according to Laplace's equations). The only charge in space exists as a surface charge on the two plates. The full solution is easiest to obtain by using the expression provided you in the previous problem:

$$\rho_v = \epsilon\frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} = \frac{\epsilon}{\rho}\frac{\partial}{\partial \phi}\left\{\frac{2V_0}{\rho\pi}\left[\frac{1}{2} - \mathrm{u}(\phi)\right]\right\} = -\frac{2\epsilon V_0}{\pi\rho^2}\delta(\phi)$$

If you got to this point, you received full credit. Note, however that $\rho_v(\vec{r}) = \rho_s(\rho,z)\frac{\delta(\phi)}{\rho}$, which is how to translate a volume charge into a surface charge. Thus, the surface charge on each sheet is

$$\rho_s = \pm\frac{2\epsilon V_0}{\pi\rho}$$

Question: ESF.6.3	*Competency:* ESF.6.3

In a uniform square-grid array of voltages that solves Laplace's equations, an unknown voltage is surrounded by the four values 20 V, 30 V, 45 V, and 65 V. What is the value of the unknown voltage?

Answer:

40 V (the average of the adjacent voltages)