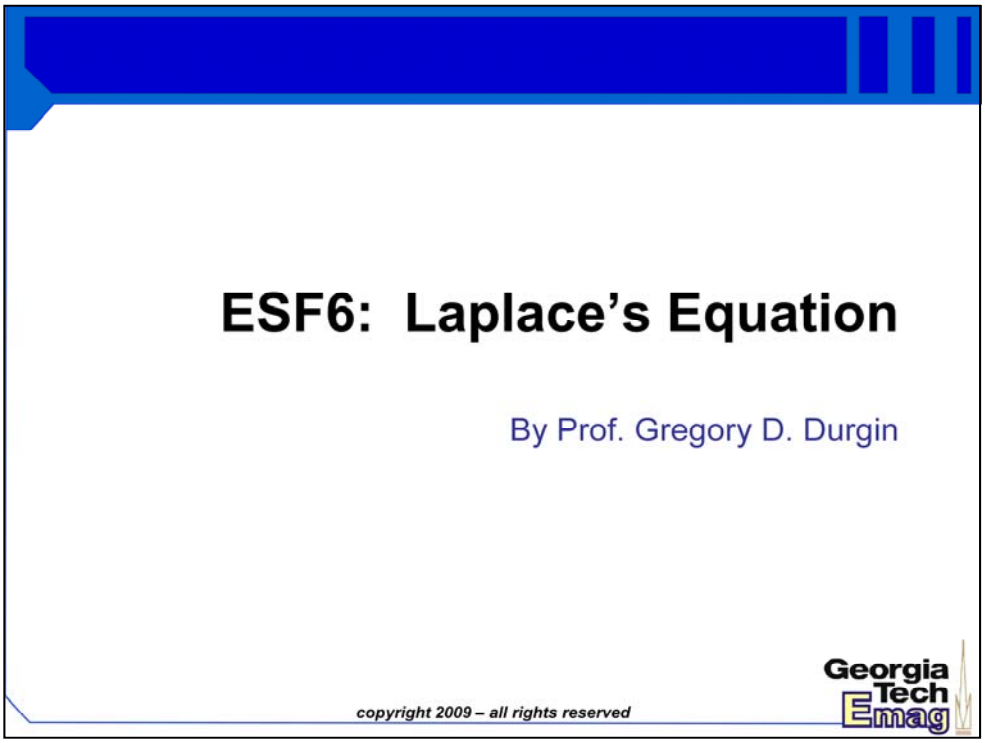
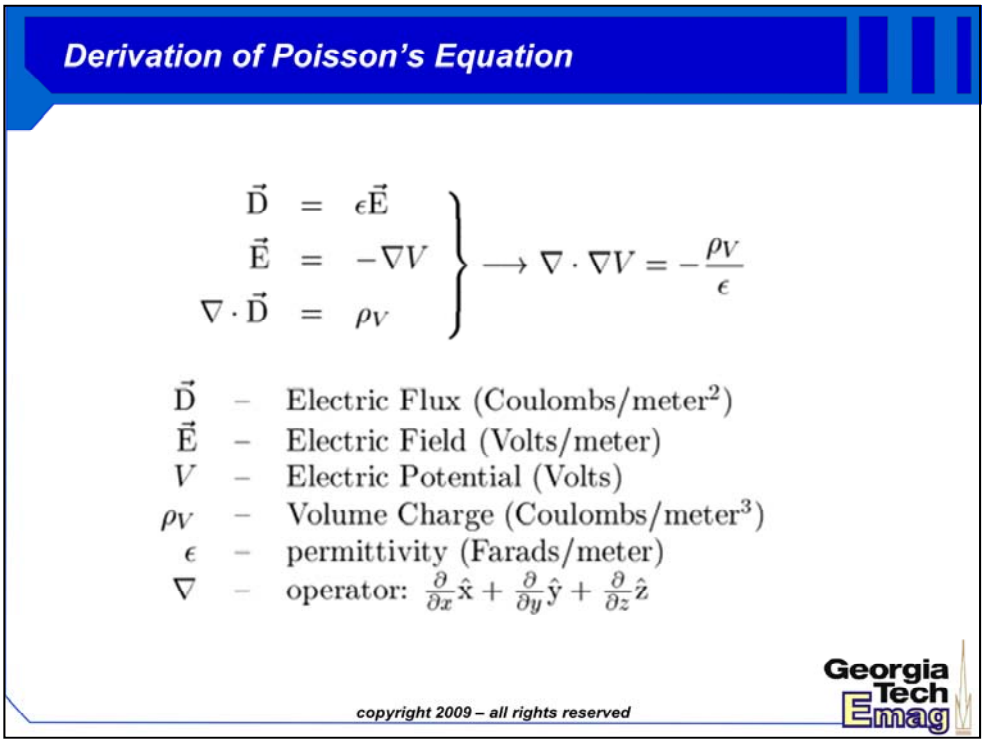


Time-harmonic analysis of physical systems is one of the most important skills for the electrical engineer to develop. Whether the application is power transmission, radio communications, data signaling, or laser emissions, the analysis of a physical system often requires physical quantities as either a single sinusoid or a superposition of multiple sinusoids. The most valuable analytical tool for studying sinusoidal physical quantities is the *phasor transform*.

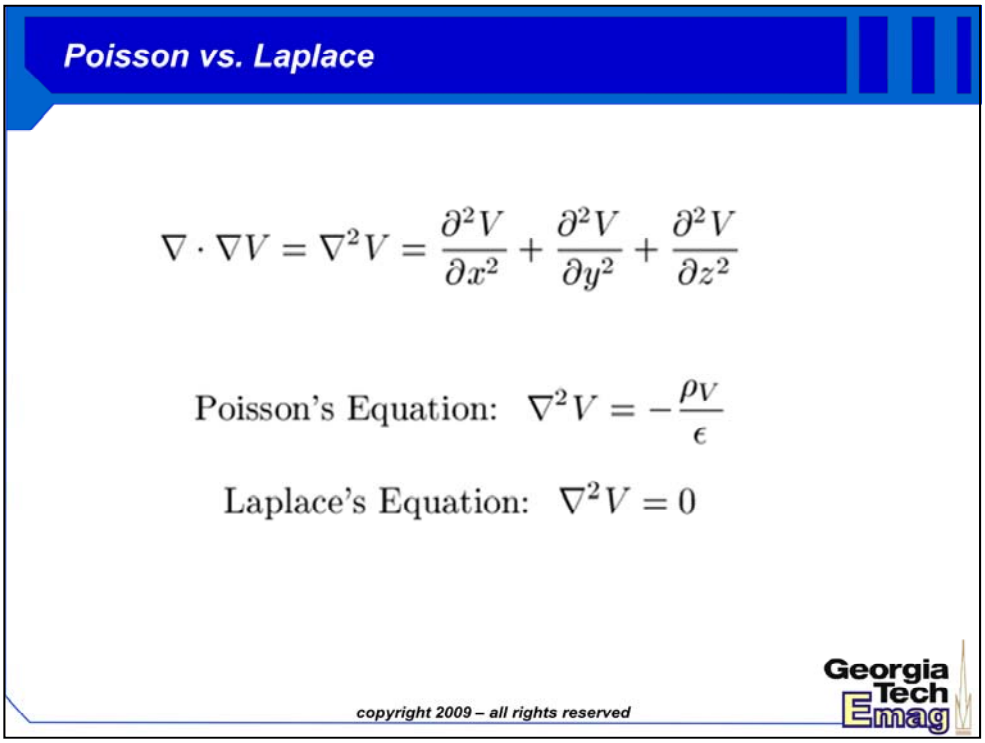
An outline of the notes on phasor transforms is given below:

- Phasors are complex numbers that are used to characterize the amplitudes and phases of time-harmonic physical quantities.
- An LTI system has the following properties:
  - The system is linear and time invariant.
  - The system is characterized by an impulse response.
  - A sinusoidal input will result in a sinusoidal output at the same frequency.



By combining our dielectric material relationship, our definition of electric potential, and Maxwell's electrostatic equation, it is possible to derive a differential equation that relates space-varying voltage to the volume charge density of space.

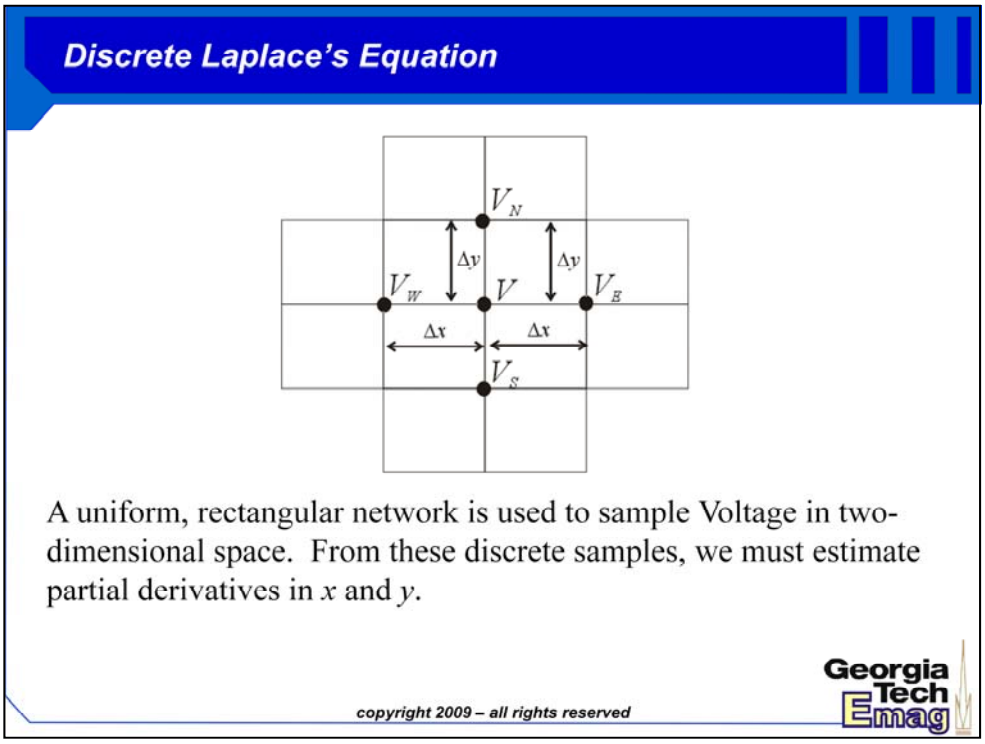
In words, this equation states the following: *the divergence of the gradient of voltage is proportional to the charge volume density at every point in space*. The operation involving the divergence of the gradient of a scalar function has a special name in the physical sciences; it is called the *Laplacian*. Thus, we could restate this equation in the following words: *the Laplacian of voltage is proportional to charge volume density*. The Laplacian operator occurs so frequently in electromagnetics and other fields that it has its own short-hand notation.



Although the Laplacian has a compact, elegant form, it defines a multivariable partial-differential equation that can be quite difficult to solve.

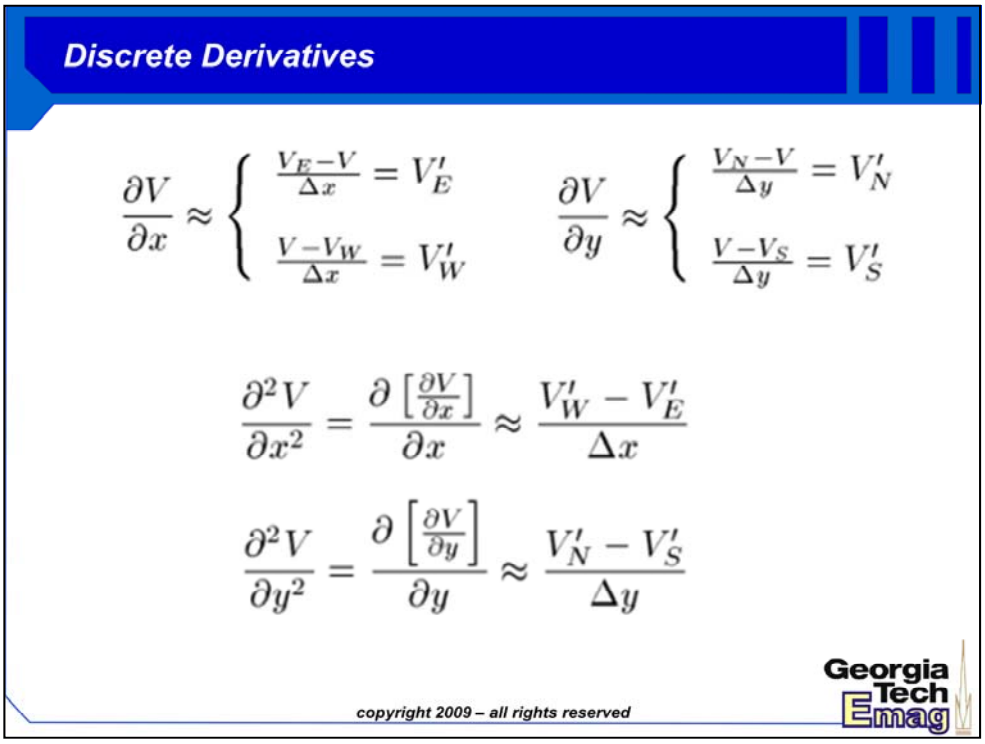
The equation that relates the Laplacian of voltage to electrostatic charge has two names, depending on the presence of charges. *Poisson's equation* is the name of this relationship when charges are present in the defined space. To solve Poisson's equation, we require two pieces of information about the solution region of space: 1) voltage boundary conditions and 2) the charge distribution. *Laplace's equation* is the name of this relationship when there are no charges present and only requires information about voltage boundary conditions. Thus, the two forms of this equation are given in the slide.

In this discussion, we will be focusing on the numerical solution of Laplace's equation, although it is very easy to extend the results to Poisson's equation.



There are very few examples of electrostatic problems that can be solved using the analytic form of Laplace's equations. Fortunately, we can recast Laplace's equation so that it is solved by a computer. This requires us to sample space, calculating the voltages in a region only at a finite number of discrete points. If we model these points accurately, then we approximate any voltage in between them through the use of *interpolation*.

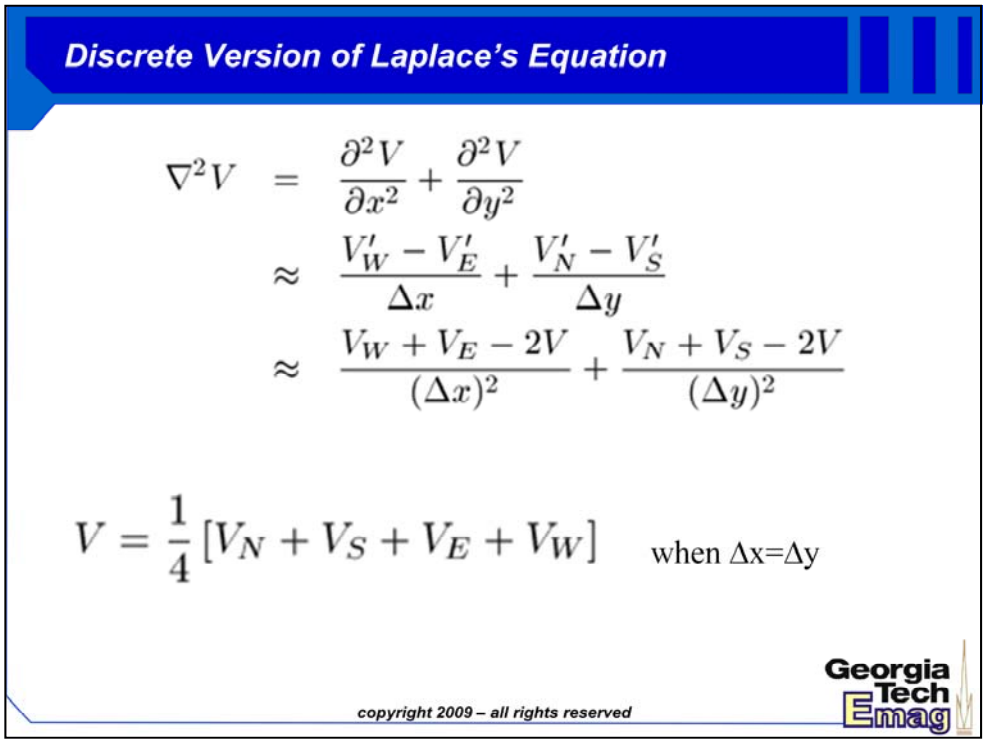
For the purposes of this discussion, we will use a rectangular grid to sample the voltage in two-dimensional space. An example of this 2D sampling is shown in Figure 1. The rectangular grid geometry is extremely easy to calculate and translate into a computer array of voltages. There are actually many other types of sampling schemes for Laplace's equation that are optimized to certain types of problems. *Finite Element Method* (FEM), for example, allows the engineer to sample space with non-uniform nodes; for FEM, regions of space that experience bigger changes in voltage receive denser samplings than regions of space with slow-varying voltages. In this way, FEM places samples in space "where they do the most good", minimizing the computation time of very large problems.



Referring to Figure 1, if we want to approximate the *first* partial derivative of voltage at a point in space, we can construct an expression based on its neighboring voltages: V_N, V_S, V_E, and V_W (North, South, East, and West). In fact, for the partial derivatives of voltage with respect to both *x* and *y*, there are two possible approximations we can use for each: (shown above)

These approximations follow very logically from the definition of a partial derivative: they mark the change in voltage in the *x* or *y* direction, divided by the spatial increment that separates the samples. Thus, we have two possible expressions for partial derivative with respect to *x* (which we will call V_E' and V_W') and two possible expressions for partial derivative with respect to *y*(which we will call V_N' and V_S').

Now that we have approximations for the first partial derivatives, we can construct approximations for the second partial derivatives as well, based on the ideal that a second partial derivative is simply a "derivative of a derivative". Thus, we can use the difference between V_E' and V_W' to estimate the second partial derivative in the *x* direction; we can use the difference between V_N' and V_S' to estimate the second partial derivative in the *y* direction. The equations for this are given above.

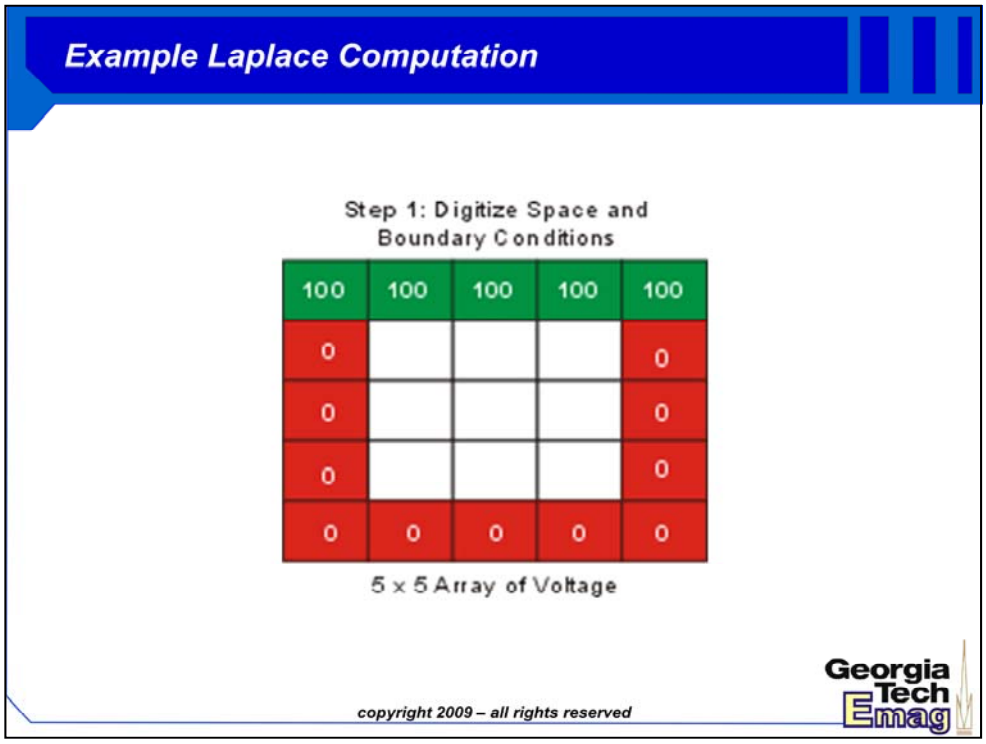


Now we have enough relationships to construct a discrete version of Laplace's equations in two dimensions. Substituting these values into the definition of the Laplacian gives us the above equation. It is this last term that, in the absence of charges in space, must be set equal to zero.

The most convenient choice of spatial increments is the case of Δx=Δy, which represents sampling on a square grid. Under these circumstances, the discrete form of Laplace's equation becomes the average of the surrounding 4 nodes.

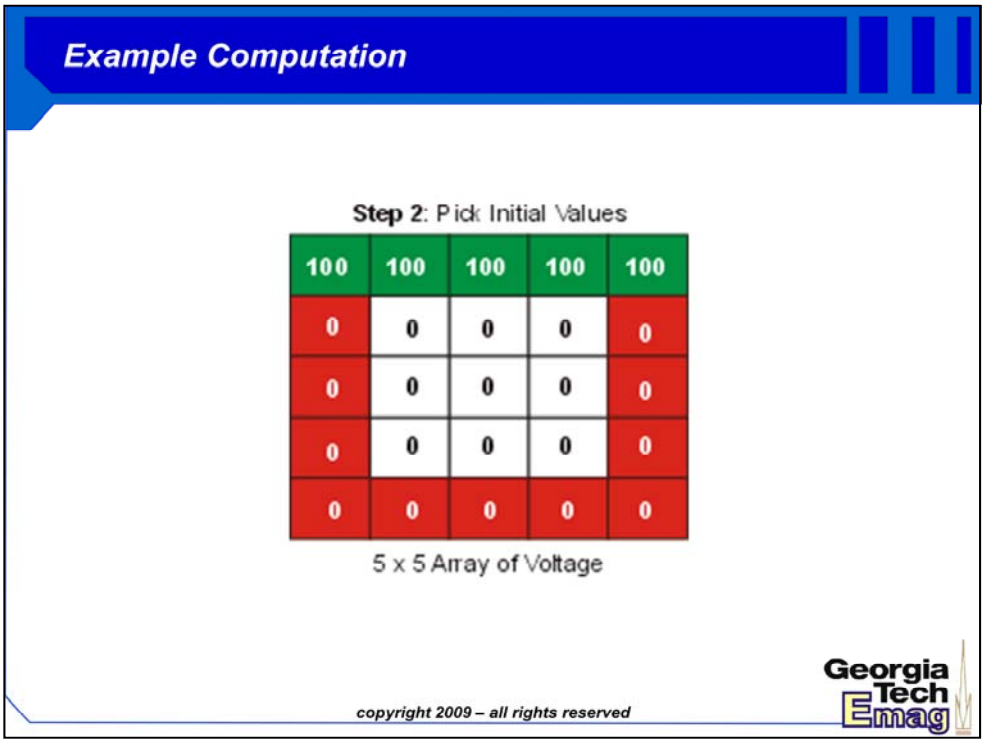
This equation is actually very simple: it states that a voltage at any particular point in uniformly sampled space must be the average of its nearest neighbors. So the discrete form of Laplace's equation is actually a 2D network of simple, interconnected averaging equations. Therefore, an *M* x *N* grid of voltage samples will produce *MN* discrete equations that can be solved iteratively by a computer.

You do not really need to have complicated computer codes to solve Laplace's equation. In fact, you can solve Laplace's equation very easily using only a spreadsheet! Simply put this averaging formula in a grid of cells, surround the cells with boundary conditions, and then iterate the calculation until the voltage values appear to converge to a final answer.



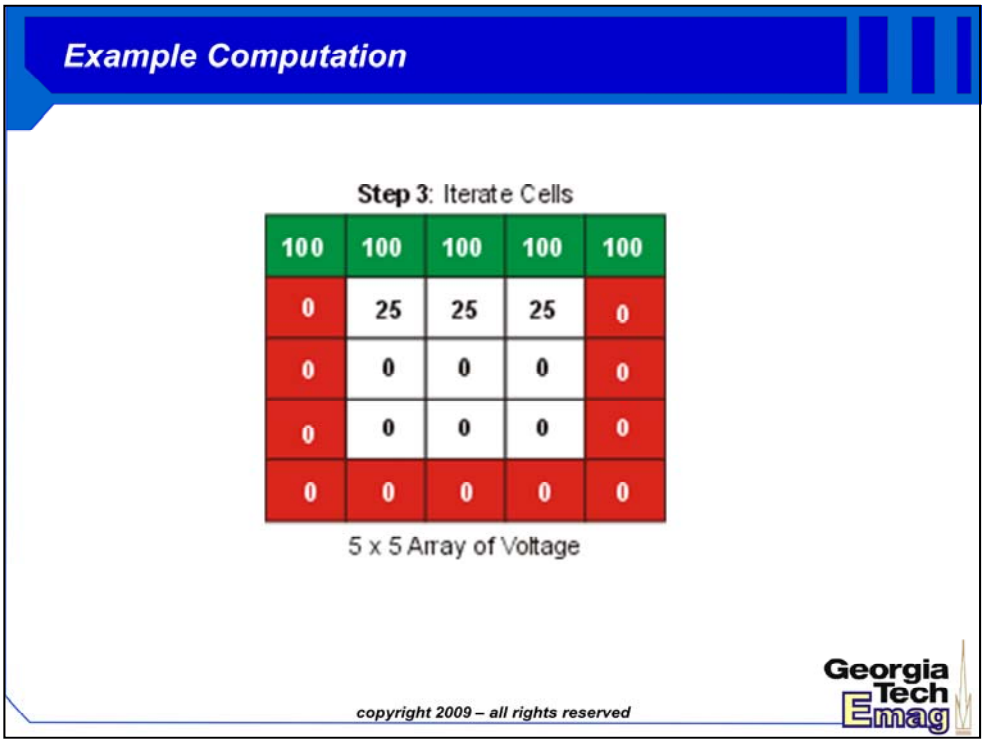
Let's work through an example of solving Laplace's equations in two dimensions. The problem that we will solve is the calculation of voltages in a square region of space. The boundary conditions for this problem involve 0 V all around the square except the top side, which is held at 100V. What is the voltage as a function of space inside the square? Our first step is to sample the square space so that we can describe it using a computer. For simplicity, we will use a 5 x 5 array of voltage values, as shown above.

This way, a 2D computer array can hold all the voltages that describe this square region of space. In this representation, all of the border elements in the matrix are set to boundary conditions; they do not change during our calculation. There are 9 unknown voltage values in the center that are not boundary conditions; these are what we will solve.



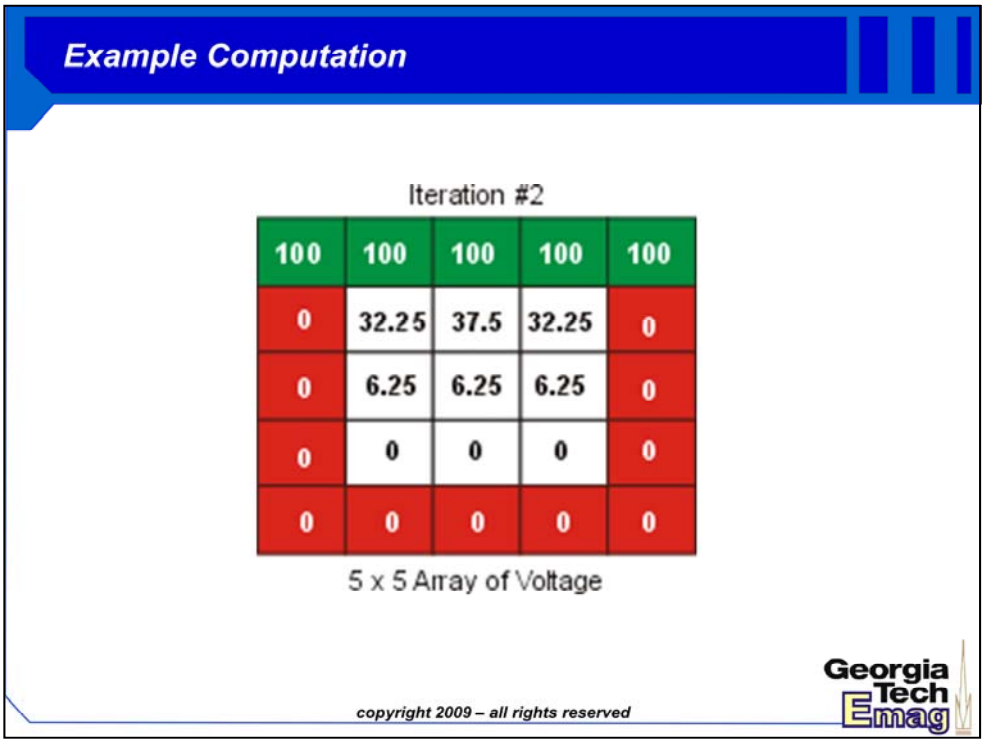
The next step is to choose initial values for the unknown voltage elements (shown in the white squares). For this example, we will set these matrix elements equal to zero.

Any initial value will work, although an intelligent choice of initial values will speed the convergence of the numerical solution. For example, this example might converge a little faster if we put a value of 50 V in the white squares (since the final voltages should be somewhere between 0 and 100 V). On the other hand, the solution for our choice of 0 V will be much faster than initial values of -4000 V or +9999 V.



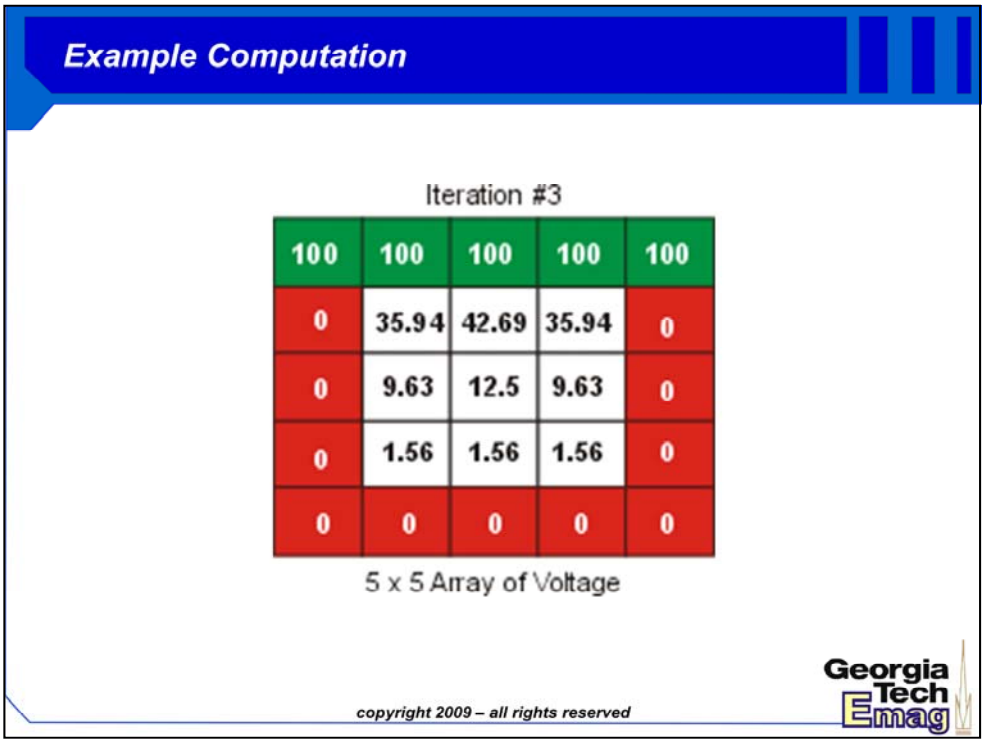
Now we iterate the equations according to a relaxation algorithm. If the medium of the square is homogeneous (i.e. the material properties do not change) then, according to the discrete form of Laplace's equation, the value in each square is equal to the average of its 4 neighboring voltages. Thus, we average all of the neighboring voltages together and place this value in the array. This is the first iteration.

Note that not much has changed. It will take several iterations before the voltages at the bottom will be non-zero. Slowly, but surely, the relaxation algorithm will converge on the physical solution.



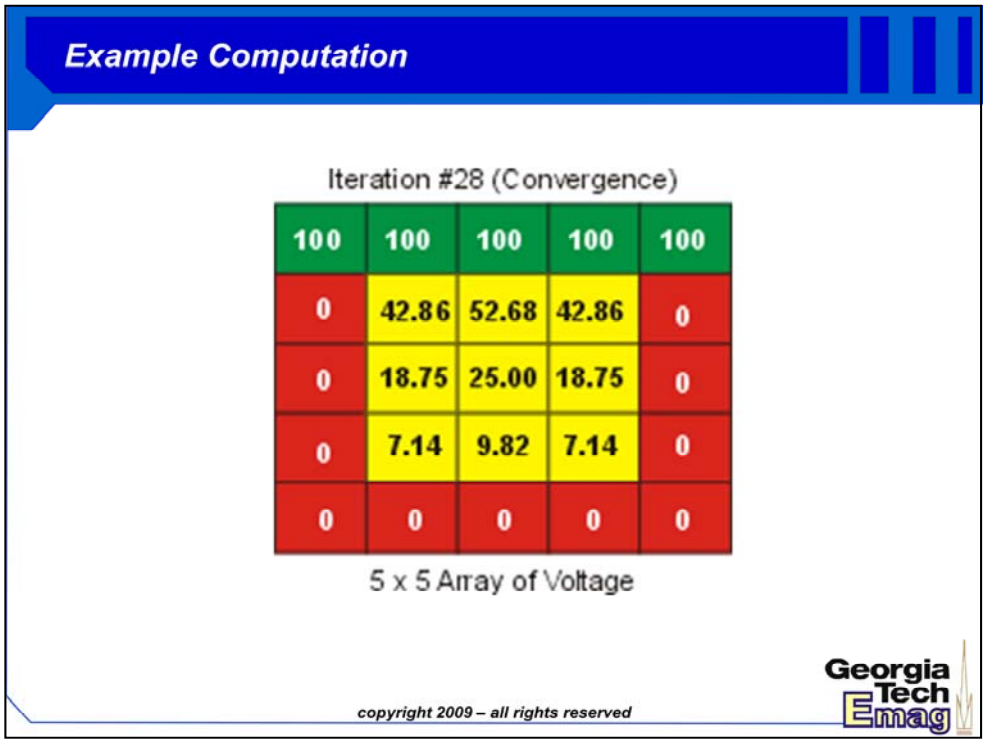
This process repeats itself. Each voltage cell is recalculated to be the average of its neighboring voltages. Iteration #2 is shown above

Example Calculation: After the first iteration was completed, the voltage in the center of the square (row 3, column 3) was surrounded by voltages of 25 (north), 0 (east), 0 (south), and 0 (west). The average of these four values is 6.25 V. Thus, this entry in the matrix receives a value of 6.25 V for iteration #2.



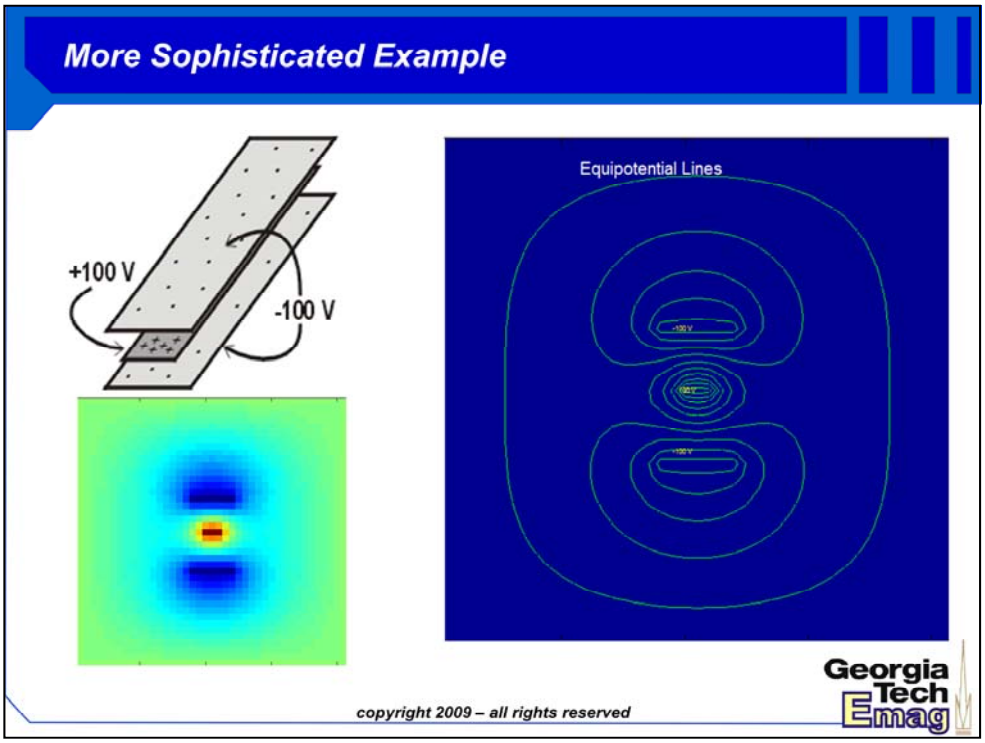
Iteration 3 continues in the same manner. The new voltages for iteration 3 are shown above.

This is starting to look more and more like a physical set of voltages. Each iteration changes the values in the matrix less and less as the solution asymptotically converges to the physical solution. In fact, this iteration process could continue indefinitely. We must have a *stopping criterion*, which is basically a test to detect whether or not we've reached the right answer. The simplest stopping criteria are based on the maximum changes in voltages from iteration to iteration. If the values in the matrix are changing very slightly, we have likely reached the solution.



We employ the following stopping criterion: we will continue to iteration *n+1*unless the largest change in any voltage from iteration *n-1* to iteration *n* was observed to be less than 0.001 V. If this is our criterion, the relaxation algorithm will stop after the 28th iteration with what should be a nearly exact representation of physical voltages in the square:

Don't let the coarseness of our matrix sampling fool you. The above voltages are very good models of electric potential at regularly spaced points within our square. We can use linear interpolation to approximate the values of voltages in between the sampled points. In this way, we can construct beautiful voltage maps and accurate equipotential lines, even though the sampling of our solution appears to be rough.



Technical specifics about this example. The center strip is held at +100V and the top and bottom strips are held at -100 V. The boundary of the region is set to 0 V. The symmetrical microstrip line (shown below) is used on multi-layer printed circuit boards to transmit high-frequency signals with lower cross-talk and radiation than regular microstrip lines.

Notice how the relaxation technique asymptotically converges to the true value. The convergence slows as the number of iterations increases.

This particular cross-section is interesting because it shows two separate "islands" of equipotential lines that grow and eventually merge. For example, there are two rings of equipotential lines for -80, -60, and -40 V -- one on the top plate and one on the bottom plate. The -20 V equipotential line, however, is a single ring around the entire cross-section.

This style of microstrip line does a good job of confining its fields to within its structure. There are very slight fringe fields around the assembly.

The final colored voltage map may look coarse and blocky, but the relaxation technique is actually very accurate. We can use some fancy rendering and interpolation between points to make a smoother map. In fact, that is what the equipotential plots have done, which is why they look so smooth.