Curriculum Topic :
Electrostatic Fields

ESF7 : Capacitance and Energy

Module Outline:	
Prerequisite Skills	Competencies
Supplemental Reading and Resources	Assessments
Power Point Slides and Notes	

Prerequisite Skills

Prerequisites / Requirements:
ESF6 Laplace's Equation

Competencies

Competency ESF.7: **Define, compute, and apply the fundamental attribute of capacitance characterizing two metal structures.**

Competency Builders:
ESF.7.1 Define capacitance between two metal structures
ESF.7.2 Setup the computation of capacitance between two metal structures
ESF.7.3 Setup the computation of per-unit-length capacitance between two traces on a transmission line.

Supplemental Reading and Resources

Supplemental Reading Materials:

Prof. Andrew Peterson's Lecture Notes (Fields and Waves Lecture 10)

Assessments

The following questions and exercises may serve as either pre-assessment or post-assessment tests to evaluate student knowledge.

Question: ESF.7.1 *Competency:* ESF.7.1

Capacitance is the ratio of ______ separation to _______ drop between two conductive structures.

Answer:

Charge, Voltage

Question: ESF.7.2 *Competency:* ESF.7.2

Capacitance of a MOSFET: An n-channel MOSFET is shown in the diagram below. It consists of metal gate, drain, and source contacts, an insulating layer of SiO_2, and an n- and p-type Si substrate. The width of the SiO_2 layer is d_1 and the cumulative width of the two Si substrates is d_2. The SiO_2 layer has permittivity ϵ_1 and the substrates both have permittivity ϵ_2. The gate has surface area (in the xy-plane) of A and is held at a potential V above ground, which is at the bottom of the substrates. Prove that the gate capacitance (with respect to the ground plane beneath the stubstrates) is given by the following expression:

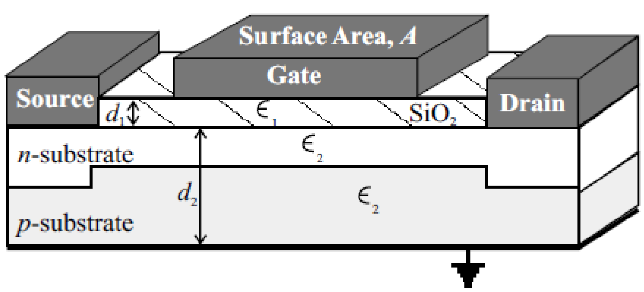
$$C = \frac{A\epsilon_1\epsilon_2}{\epsilon_2 d_1 + \epsilon_1 d_2}$$

Hint: The definition of capacitance is $C = Q/V$. You may approximate this scenario as a parallel-plate capacitor where the electric flux density underneath the gate is given by

$$\vec{D} = -\rho_S \hat{a}_z$$

where ρ_S is the surface charge density on the gate.

n-Channel Depletion MOSFET



Answer:

Capacitance of a MOSFET: We use the definition $C = Q/V$. If there is a charge density of $+\rho_S$ on the gate, then the total charge is $Q = A\rho_S$. If the flux density beneath the gate is $-\rho_S\hat{a}_z$, then the fields in each region are:

$$\text{SiO}_2 : \vec{E}_1 = -\frac{\rho_S}{\epsilon_1}\hat{a}_z \qquad \text{Si Substrates} : \vec{E}_2 = -\frac{\rho_S}{\epsilon_2}\hat{a}_z$$

Recall that normal $\vec{D}$ components are equal across dielectric boundaries. To get voltage, we simply integrate these fields from bottom to top:

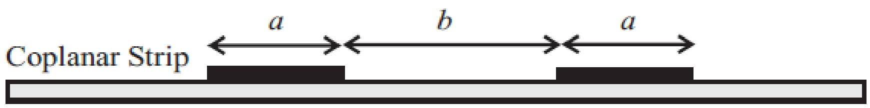
$$\begin{aligned} V &= -\int_A^B \vec{E}\cdot d\vec{L} \\ &= -\int_0^{d_2} \vec{E}_2 \cdot dz\hat{a}_z - \int_{d_2}^{d_1+d_2} \vec{E}_1 \cdot dz\hat{a}_z \\ &= \rho_S\left[\frac{d_1}{\epsilon_1} + \frac{d_2}{\epsilon_2}\right] \end{aligned}$$

Thus,

$$C = \frac{Q}{V} = \frac{A\rho_S}{\rho_S\left[\frac{d_1}{\epsilon_1}+\frac{d_2}{\epsilon_2}\right]} = \frac{A\epsilon_1\epsilon_2}{\epsilon_2 d_1 + \epsilon_1 d_2}$$

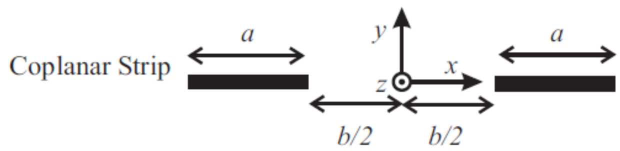
Question: ESF.7.3 *Competency:* ESF.7.3

Bonus Challenge: Use these results to approximate the per-unit-length capacitance of a coplanar strip transmission line with trace width a and separation distance b. (+5 points)

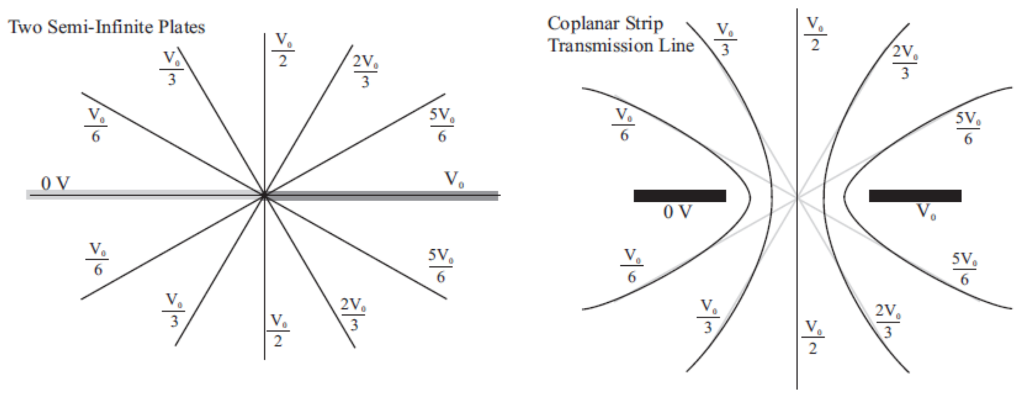


Answer:

(d) **Bonus Challenge:** No one got this. Here is the procedure, for those interested. First, note that we should adapt the geometry of our simple charge plates to the geometry of the transmission line:



Compare a sketch of equipotential lines below for the two semi-infinite plates that you solved in this test problem to the coplanar strip transmission line with finite plates:



Your expressions are a pretty good approximation to the fields and charges in this very practical problem. The approximation only breaks down close to the origin. However, many engineers use the approximations of this problem to derive per-unit-length capacitance and inductance for coplanar strip transmission lines – particularly when b is small relative to a. If this is true, then per-unit-length capacitance is given by

$$C = \frac{\rho_L}{V_0} = \frac{1}{V_0} \int_{\frac{b}{2}}^{\frac{b}{2}+a} \rho_s \, d\rho = \int_{\frac{b}{2}}^{\frac{b}{2}+a} \frac{2\epsilon V_0 \, d\rho}{\pi \rho} = \frac{2\epsilon}{\pi} \ln\left(1 + \frac{2a}{b}\right)$$

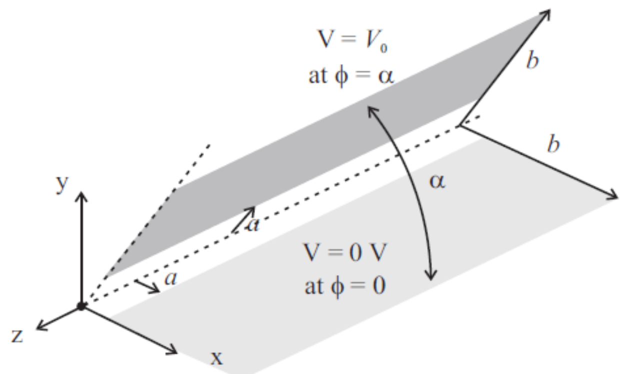
Question: ESF.7.4 *Competency:* ESF.7.3

Wedge Transmission Line: Two long conductive plates run the length of the z-axis, one lying in the $\phi = 0$ plane and the other lying in the $\phi = \alpha$ plane. Their leading edges are each a units from the z-axis and their furthest edges are each b units from the z-axis. The plates are held at 0 and V_0 Volts as shown in the diagram below. When $b >> a$, the voltage in this homogeneous region of space is given by the following solution:

$$V(\rho,\phi,z) = \begin{cases} \frac{V_0\phi}{\alpha} & 0<\phi<\alpha \\ \frac{V_0}{2\pi-\alpha}[2\pi-\phi] & \alpha<\phi<2\pi \end{cases} \quad \textbf{or} \quad V_0\left[\frac{\phi}{\alpha} - \left[\frac{2\pi}{\alpha(2\pi-\alpha)}\right](\phi-\alpha)\mathrm{u}(\phi-\alpha)\right]$$

where $\mathrm{u}(\cdot)$ is the unit step function. Answer the following questions based on this scenario.

(a) Verify that this equation satisfies Laplace's equation, $\nabla^2 V = 0$, in all regions between the plates ($0 < \phi < \alpha$ and $\alpha < \phi < 2\pi$) that contain no charge. (10 points)



(b) In the space below, show that the electric field distribution in space is the following: (10 points)

$$\vec{E}(\rho,\phi,z) = \begin{cases} -\frac{V_0}{\rho\alpha}\hat{\phi} & 0<\phi<\alpha \\ \frac{V_0}{\rho(2\pi-\alpha)}\hat{\phi} & \alpha<\phi<2\pi \end{cases} \quad \textbf{or} \quad -\frac{V_0}{\rho\alpha}\left[1 - \frac{2\pi}{2\pi-\alpha}\mathrm{u}(\phi-\alpha)\right]\hat{\phi}$$

(c) Estimate the per-unit-length capacitance of this transmission line. (20 points) Hint: the following relationship between surface charge density and volume charge density in cylindrical coordinates may prove useful for the top plate: $\rho_v(\vec{r}) = \rho_s(\rho, z)\frac{\delta(\phi-\alpha)}{\rho}$.

Answer:

(a) Taking the Laplacian of voltage in both regions of space:

$$\nabla^2 V = \begin{cases} \nabla^2 \frac{V_0 \phi}{\alpha} & 0<\phi<\alpha \\ \nabla^2 \frac{V_0}{2\pi - \alpha}\left[2\pi - \phi\right] & \alpha<\phi<2\pi \end{cases}$$

$$= \begin{cases} \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \phi}\frac{V_0 \phi}{\alpha}\right\} & 0<\phi<\alpha \\ \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \phi}\frac{V_0}{2\pi - \alpha}\left[2\pi - \phi\right]\right\} & \alpha<\phi<2\pi \end{cases}$$

$$= \begin{cases} \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\frac{V_0}{\alpha} & 0<\phi<\alpha \\ \frac{1}{\rho^2}\frac{\partial}{\partial \phi}\frac{-V_0}{2\pi - \alpha} & \alpha<\phi<2\pi \end{cases}$$

$$= \begin{cases} 0 & 0<\phi<\alpha \\ 0 & \alpha<\phi<2\pi \end{cases}$$

which agrees with the chargeless region of space around the metal plates.

(b) Recognize that $\vec{E} = -\nabla V(\phi)$. The key difficulty in the problem is to make sure we use the gradient formula for the *cylindrical* coordinate system:

$$-\nabla V(\phi) = \begin{cases} -\nabla \frac{V_0 \phi}{\alpha} & 0<\phi<\alpha \\ -\nabla \frac{V_0}{2\pi - \alpha}\left[2\pi - \phi\right] & \alpha<\phi<2\pi \end{cases}$$

$$= \begin{cases} -\frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial \phi}\frac{V_0 \phi}{\alpha} & 0<\phi<\alpha \\ -\frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial \phi}\frac{V_0}{2\pi - \alpha}\left[2\pi - \phi\right] & \alpha<\phi<2\pi \end{cases}$$

$$= \begin{cases} -\frac{V_0}{\rho\alpha}\hat{\phi} & 0<\phi<\alpha \\ \frac{V_0}{\rho(2\pi - \alpha)}\hat{\phi} & \alpha<\phi<2\pi \end{cases}$$

which is the stated solution for E-field.

(c) To find charge from an electric field distribution, we apply Gauss's law in differential form: $\nabla \cdot \vec{D} = \rho_v$. Note that the divergence formula in cylindrical coordinates predicts that

$$\nabla \cdot (\epsilon \vec{E}) = \epsilon \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} = -\frac{\epsilon V_0}{\rho^2 \alpha}\frac{\partial}{\partial \phi}\left[1 - \frac{2\pi}{2\pi - \alpha}\mathrm{u}(\phi - \alpha)\right] = \frac{\epsilon V_0}{\rho^2 \alpha}\frac{2\pi}{2\pi - \alpha}\delta(\phi - \alpha)$$

This result makes intuitive sense, confirming what we learned from part (a) that there is no charge in the region around the plates. For the region $\phi = \alpha$, however, the delta-function is non-zero, indicating the presence of surface charge on this plane. In fact, given the hint that $\rho_v(\vec{r}) = \rho_s(\rho, z)\frac{\delta(\phi - \alpha)}{\rho}$ (a result easily derived from the calculus of cylindrical coordinates) we can see that the top plate must contain the following surface charge distribution:

$$\rho_s(\rho, z) = +\frac{\epsilon V_0}{\rho\alpha}\left(\frac{2\pi}{2\pi - \alpha}\right)$$

which gradually tapers the otherwise uniform charge for locations away from the z-axis. The $\phi = 0$ plate would contain the negative counterpart of this charge.

Now we have enough information to estimate the per-unit-length capacitance. Calculating the charge per unit length along the z-axis:

$$\rho_L = \int_a^b \rho_s \, d\rho = \frac{\epsilon V_0}{\alpha} \frac{2\pi}{2\pi - \alpha} \int_a^b \frac{d\rho}{\rho} = \frac{\epsilon V_0}{\alpha} \frac{2\pi}{2\pi - \alpha} \ln\left(\frac{b}{a}\right)$$

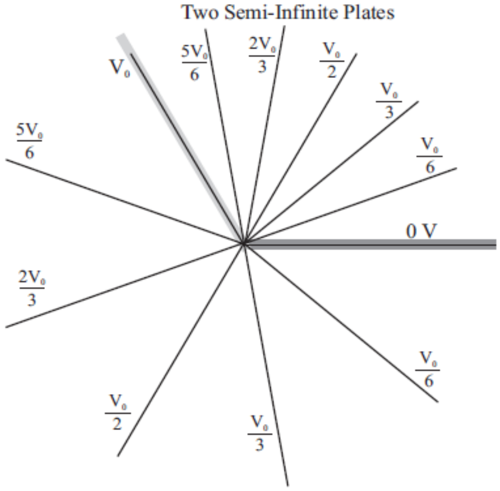
Plugging this result into our definition for per-unit-length capacitance produces

$$C = \frac{\rho_L}{V_0} = \frac{2\pi\epsilon}{\alpha(2\pi - \alpha)} \ln\left(\frac{b}{a}\right)$$

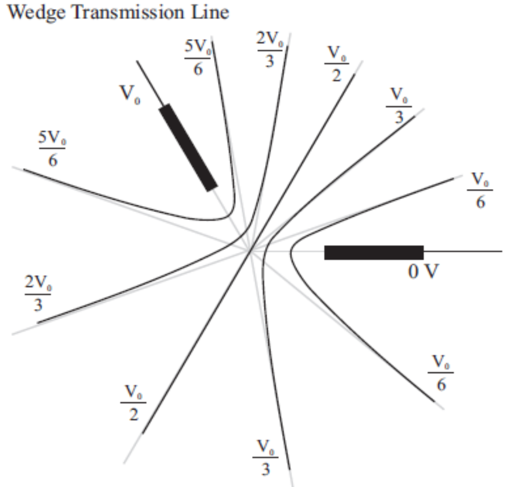
Does this answer make sense? Yes, because it exhibits all of the intuitive behavior we might expect of capacitance. As the surface area of the plates gets larger (the ration of b/a), capacitance increases. There is a local minimum for plates when they are bent farthest apart ($\alpha = \pi$); capacitance increases as the plates are brought closer together.

As mentioned in the problem statement, this field expression is somewhat approximate (it is exact only as $b \longrightarrow \infty$ and $a \longrightarrow 0$). Consider a cross-sectional sketch of equipotential lines for the case of the ideal semi-infinite wedge and the finite wedge:

Two Semi-Infinite Plates



Wedge Transmission Line



Compared side by side, the solutions are almost identical except close-in to the z-axis. However, the solution is still an excellent approximation around the plates and useful for estimating surface charges.