ESF7: Capacitance and Energy

By Prof. Gregory D. Durgin

Georgia Tech Emag

Recap of Electrostatics

4 Physical Quantities

Q - charge (Coulombs), ρ_v, ρ_s, ρ_L

$\vec{E}$ - electric field (V/m), proportional to force on charges $\vec{F} = Q\vec{E}$

V - electric potential. Scalar function, representing potential to do work

$\vec{D}$ - electric flux (density), carrier of charge effects through space. C/m^2

Georgia Tech Emag

Toolbox for Electrostatic Analysis

You Know	You Want	Eqn
Charge (at $\vec{r}'$)	$\vec{E}(\vec{r})$	$\vec{E} = \dfrac{Q(\vec{r}-\vec{r}')}{4\pi\epsilon\|\vec{r}-\vec{r}'\|^3}$ or $\iiint_{vol} \dfrac{\rho_v(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\epsilon\|\vec{r}-\vec{r}'\|^3} dv$
Charge	$V(\vec{r})$	$V = \dfrac{Q}{4\pi\epsilon\|\vec{r}-\vec{r}'\|}$ or $= \int_{vol} \dfrac{\rho_v(\vec{r}')}{4\pi\epsilon\|\vec{r}-\vec{r}'\|} dv$
$\vec{E}(\vec{r})$	$\vec{D}(\vec{r})$	$\vec{D} = \epsilon\vec{E}$
$\vec{D}$	Charge ρ_v	$\rho_v = \nabla\cdot\vec{D}$; $\oiint \nabla\cdot\vec{D} = \int_{vol} \rho_v(\vec{r}')\,dV$

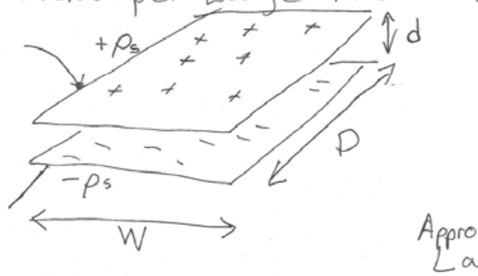
Definition of Capacitance

$$C = \frac{Q}{V} \qquad \frac{\text{colombs}}{\text{volts}}$$

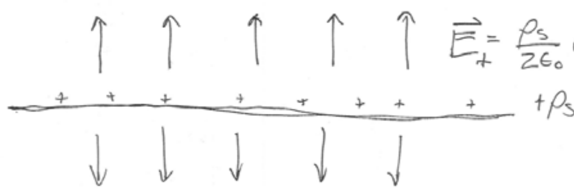
definition of capacitance

Parallel Plate Capacitance

Fields per Large Plates (air filled)



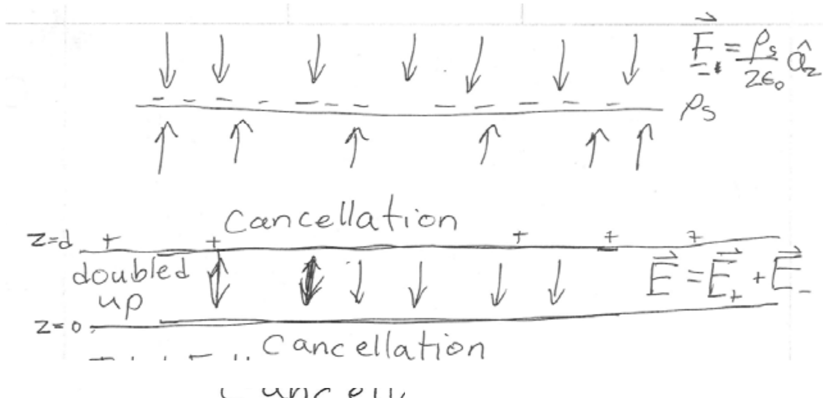
Approximate as Large



$$\vec{E}_+ = \frac{\rho_s}{2\epsilon_0}\hat{a}_z$$

Georgia Tech Emag

Field Solution of Parallel Plates



Total Field:

$$\vec{E} = \frac{\rho_s}{\epsilon_0}\hat{a}_z$$

Georgia Tech Emag

Parallel Plate Voltage Calculation

Total Field:

$$\vec{E} = \frac{\rho_s}{\epsilon_o}\hat{a}_z \qquad d\vec{L} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$

$$V = -\int_0^d \vec{E}\cdot d\vec{L} = +\int_0^d \frac{\rho_s}{\epsilon_o}\,dz$$

$$= \frac{\rho_s d}{\epsilon_o}$$

Capacitance of Parallel Plates

Total Charge:

$$Q = W D \rho_s \qquad (\text{length} \times \text{width}) \times \frac{\text{Coulombs}}{m^2}$$

$$C = \frac{Q}{V} = \frac{W D \rho_s}{\frac{\rho_s d}{\epsilon_o}} = \frac{W D \epsilon_o}{d}$$

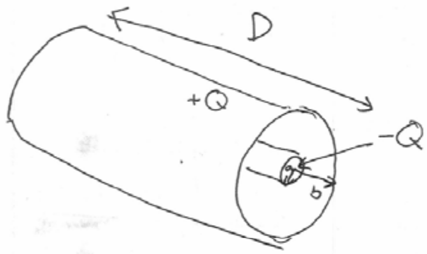
per unit length capacitance

$$C = \frac{D}{d}\epsilon_o$$

remember this from transmission lines.

Capacitance of a Coaxial Structure

Coaxial "Capacitor"



Result

$$E_\rho = \frac{\rho_L}{2\pi\rho\epsilon}$$

$$\rho_L = \int_0^{2\pi}\int_0^{R} \rho_v(\rho)\,\rho\, d\rho\, d\phi$$

for infinite charges with radial symmetry independent of z

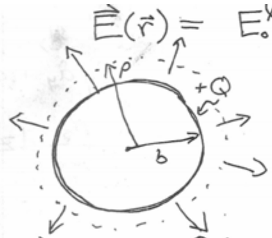
$$\vec{E}(\vec{r}) = E_0^{(\rho)}\hat{a}_\rho$$

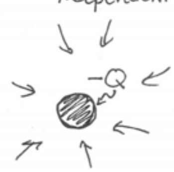
Georgia Tech Emag

Solving for Coaxial Electric Field

$$\vec{E}(\vec{r}) = E_0^{(\rho)}\hat{a}_\rho$$

with radial symmetry independent of z





$$\vec{E}_o(\rho) = \begin{cases} \dfrac{Q/D}{2\pi\rho\epsilon} & \rho \geq b \\ 0 & \rho \leq b \end{cases}$$

$$\vec{E}_i(\rho) = \begin{cases} \dfrac{-Q/D}{2\pi\rho\epsilon} & \rho \geq a \\ 0 & \rho < a \end{cases}$$

$$\vec{E}_{total} = \begin{cases} 0 & \rho \geq b \\ \dfrac{-Q}{2\pi\rho\epsilon D} & a \leq \rho \leq b \\ 0 & \rho < a \end{cases}$$

Georgia Tech Emag

Coaxial Capacitance

$$V = -\int_a^b \vec{E}\cdot d\vec{L}$$

$$= -\int_a^b \left(\frac{-Q\hat{\rho}}{2\pi\rho\epsilon D}\right)\cdot d\rho\,\hat{\rho}$$

$$= \frac{Q}{2\pi\epsilon D}\int_a^b \frac{d\rho}{\rho}\,\overbrace{(\hat{\rho}\cdot\hat{\rho})}^{1}$$

$$= \frac{Q}{2\pi\epsilon D}\ln\rho\Big|_a^b = \frac{Q}{2\pi\epsilon D}\ln\frac{b}{a}$$

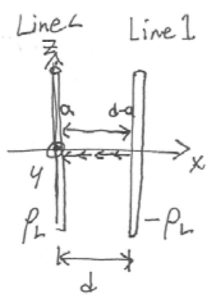
$$C = \frac{Q}{V} = \frac{2\pi\epsilon D}{\ln b/a}$$

per-unit-length capacitance

$$C' = \frac{C}{D} = \frac{2\pi\epsilon}{\ln b/a}$$ our T-line formula!

- Summary of steps:
  - Calculate field from charge separation
  - Calculated voltage from field
  - Divide voltage into charge separation
- Derivation of our original transmission line formula

Example of Capacitance Between Wires



Capacitance of Two Wires

$$\vec{E}(\rho) = \frac{\rho_L}{2\pi\epsilon\rho}\,\hat{\rho}$$ for line charge on z-axis

$$\vec{E}(x,0,0) = \frac{\rho_L}{2\pi\epsilon x}\,\hat{x} - \frac{\rho_L}{2\pi\epsilon(x-d)}\,\hat{x}$$

Two-Wire Solution

$$V_{12} = -\int_{d-a}^{a} \vec{E}(x,0,0) \cdot \underbrace{\hat{x}\,dx}_{d\vec{\ell}}$$

$$= \frac{\rho_L}{2\pi\epsilon}\int_a^{d-a} \frac{dx}{x} - \frac{\rho_L}{2\pi\epsilon}\int_a^{d-a} \frac{dx}{x-d}$$

$$= \frac{\rho_L}{2\pi\epsilon}\left[\ln x \Big|_a^{d-a} - \ln(x-d)\Big|_a^{d-a} \right]$$

$$= \frac{\rho_L}{2\pi\epsilon}\left[\ln\frac{d-a}{a} - \underbrace{\ln\frac{-a}{a-d}}_{\ln\frac{a}{d-a} = -\ln\frac{d-a}{a}} \right]$$

$$= \frac{\rho_L}{\pi\epsilon}\ln\left(\frac{d-a}{a}\right)$$

$$= \frac{\rho_L}{\pi\epsilon}\ln\left(\frac{d}{a}-1\right)$$

$$C = \frac{\rho_L}{V}$$

$$= \frac{\pi\epsilon}{\ln\left(\frac{d}{a}-1\right)} \quad F/m$$

- Approximate solution for significant wire separation (d>>a)
- For small a, charge on wire gravitates asymmetrically to the side closer the other wire

Georgia Tech Emag

Work in a System with Two Charges

Work, for 2 point Charges

Q_1 Q_2 $V(\vec{r})$

$$W_E = Q_2 V_{2,1} \quad \text{or} \quad Q_1 V_{1,2}$$

$V_{x,y}$ is Voltage on x due to y.

Georgia Tech Emag

Work for Multiple Charges, Charge Distributions

$$W = \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} Q_m V_{m,n}$$

$$= \frac{1}{2}\sum_{m=1}^{N} Q_m \underbrace{\sum_{n=1}^{N} V_{m,n}}_{V_m}$$

$$= \frac{1}{2}\sum_{m=1}^{N} Q_m V_m$$

$$\boxed{W = \frac{1}{2}\iiint_{vol} \rho_v V \, d\mathcal{V}}$$

Georgia Tech Emag

Work as a Function of Electric Field

We know: $\rho_v = \nabla \cdot \vec{D}$

$$W = \frac{1}{2}\iiint_{vol} (\nabla \cdot \vec{D}) V \, dv$$

Vector Calc Relationship

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$$

Georgia Tech Emag

Vector Calculus Manipulations

Vector Calc Relationship

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$$

$$W = \frac{1}{2} \iiint_{Vol} \left[\nabla \cdot (V\vec{D}) - \vec{D} \cdot (\nabla V)\right] dV$$

$$= \frac{1}{2} \oint_S (V\vec{D}) \cdot d\hat{n} + \frac{1}{2} \int_{Vol} \vec{D} \cdot \underbrace{(-\nabla V)}_{\vec{E}} dv$$

Work as a Function of Electric Field

Let us say that volume is arbitrarily large and contains all charges that create fields.

Then first integral is 0 and

$$\boxed{W = \frac{1}{2} \int_{Vol} \vec{D} \cdot \vec{E} \, dv}$$

$$W = \frac{1}{2} \epsilon_0 \int_{Vol} |\vec{E}|^2 dV$$

Sum of magnitude of all fields