TDT2: Transmission Line Equations

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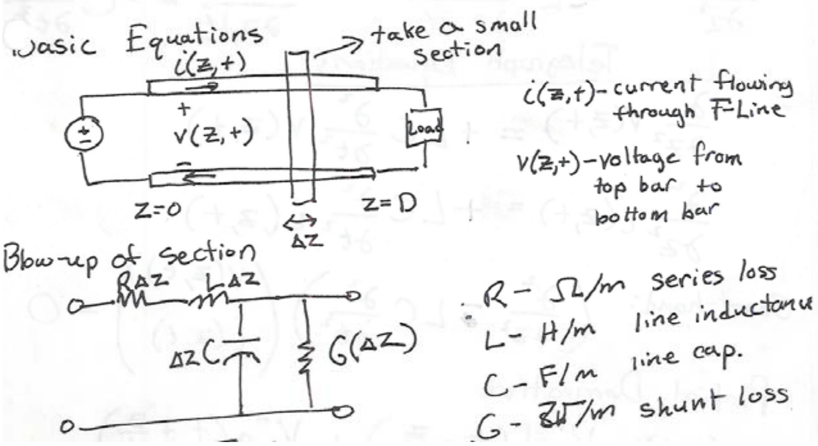
Circuit Model for a Transmission Line

Basic Equations



$i(z,t)$ - current flowing through T-Line

$v(z,t)$ - voltage from top bar to bottom bar

R - Ω/m series loss

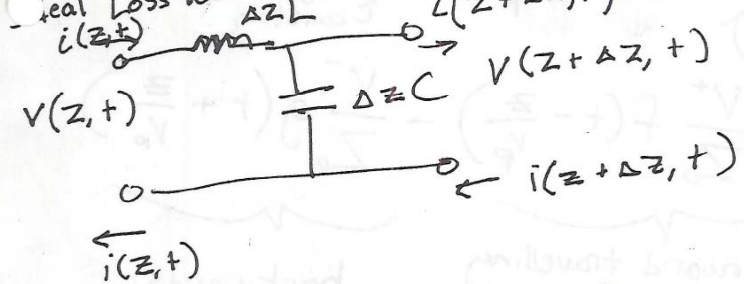
L - H/m line inductance

C - F/m line cap.

G - $\mho$/m shunt loss

Segment Circuit Model

Ideal Loss less T-Line



$$-v(z+\Delta z,t)+v(z,t)=\Delta z\, L\, \frac{\partial i(z,t)}{\partial t}$$

$$-i(z+\Delta z,t)+i(z,t)=\Delta z\, C\, \frac{\partial v(z,t)}{\partial t}$$

The Telegrapher's Equations

Limit $\Delta z \rightarrow 0$

$$\frac{\partial v(z,t)}{\partial z} = -L\frac{\partial i(z,t)}{\partial t} \qquad \frac{\partial i(z,t)}{\partial z} = -C\frac{\partial v(z,t)}{\partial t}$$

$$\frac{\partial^2 v(z,t)}{\partial z^2} = L\frac{\partial^2 i(z,t)}{\partial z\,\partial t} \qquad \frac{\partial^2 i(z,t)}{\partial z\,\partial t} = -C\frac{\partial^2 v(z,}{\partial t^2}$$

Telegraph Equations

$$\frac{\partial^2}{\partial z^2}v(z,t) = +LC\frac{\partial^2}{\partial t^2}v(z,t)$$

$$\frac{\partial^2}{\partial z^2}i(z,t) = +LC\frac{\partial^2}{\partial t^2}i(z,t)$$

Short-hand: $\left(\frac{\partial^2}{\partial z^2} - LC\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix} v(z,t) \\ i(z,t)\end{pmatrix} = 0$

Solution to the Telegrapher's Equations

Partial Derrivative

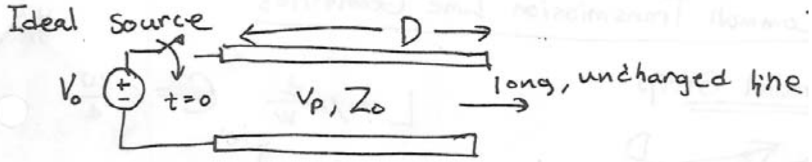
$$v(z,t) = \underset{\text{constant}}{V^+} f\left(t - \frac{z}{v_p}\right) + \underset{\text{constang}}{V^-} g\left(t + \frac{z}{v_p}\right)$$

$$i(z,t) = \underbrace{\frac{V^+}{Z_o} f\left(t - \frac{z}{v_p}\right)}_{\text{forward travelling wave}} - \underbrace{\frac{V^-}{Z_o} g\left(t + \frac{z}{v_p}\right)}_{\text{backwards travelling wave}}$$

$$Z_o = \sqrt{\frac{L}{C}}$$

$$v_p = \frac{1}{\sqrt{LC}}$$

The shape of the functions f and g are determined by boundary conditions on the line. (Sources and loads connected to the line.)



excitation: $V_o u(t) = f(t)$

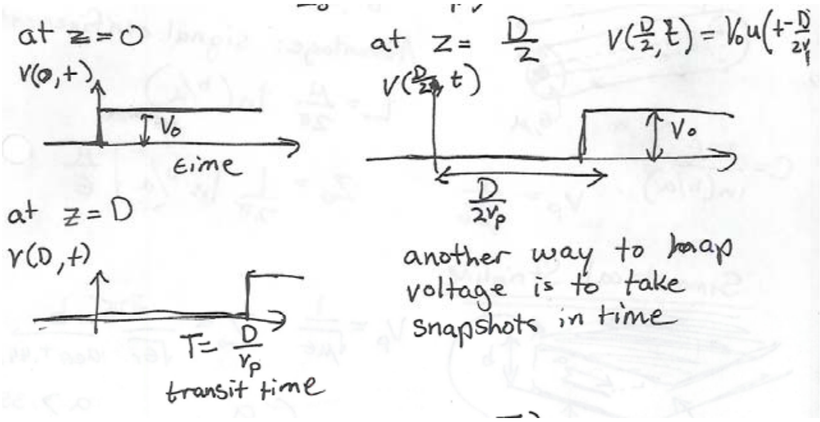
boundary condition excitation $= v(0,t)$ on line

$$v(z,t) = V_o u\left(t - \frac{z}{v_p}\right)$$

$$i(z,t) = \frac{V_o}{Z_o} u\left(t - \frac{z}{v_p}\right)$$

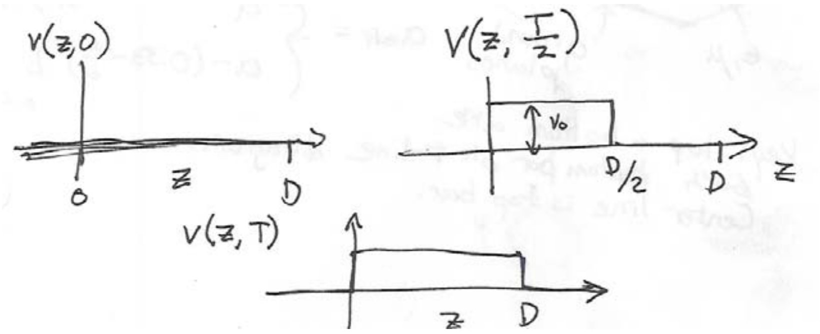
at $z = 0$

Movement of a Pulse Over Time



another way to map voltage is to take snapshots in time

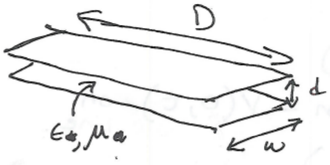
Movement of a Pulse Over Space



Parallel Strip Transmission Lines

Common Transmission Line Geometries

Parallel Strip



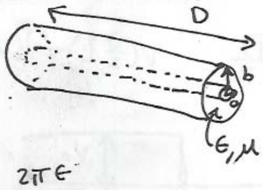
$$L = \mu \frac{d}{w} \qquad C = \epsilon \frac{w}{d}$$

$$V_p = \frac{1}{\sqrt{\mu\epsilon}}$$

$$Z_o = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}}$$

Coaxial Cable

Coaxial Cable



a – inner radius
b – outer radius

Advantage: signal confinement

$$L = \frac{\mu}{2\pi} \ln(b/a)$$

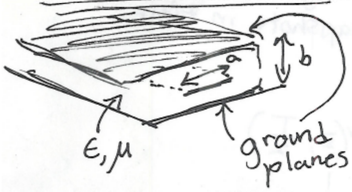
$$C = \frac{2\pi\epsilon}{\ln(b/a)}$$

$$V_p = \frac{1}{\sqrt{\mu\epsilon}}$$

$$Z_o = \frac{1}{2\pi} \ln b/a \sqrt{\frac{\mu}{\epsilon}}$$

Symmetrical Stripline

Symmetrical Stripline



$$V_p = \frac{1}{\sqrt{\mu\epsilon}} \quad Z_o = \frac{30\pi}{\sqrt{\epsilon_r}} \frac{b}{a_{eff} + .441b}$$

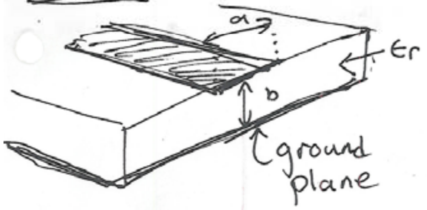
$$a_{eff} = \begin{cases} a & a > .35b \\ a - (0.35 - \frac{a}{b})^2 b & a < 0.35 \end{cases}$$

Key: top + bottom are both ~~bottom bar~~ on t-line diagram.
Center line is top bar.

Microstrip Line

Microstrip

5/5

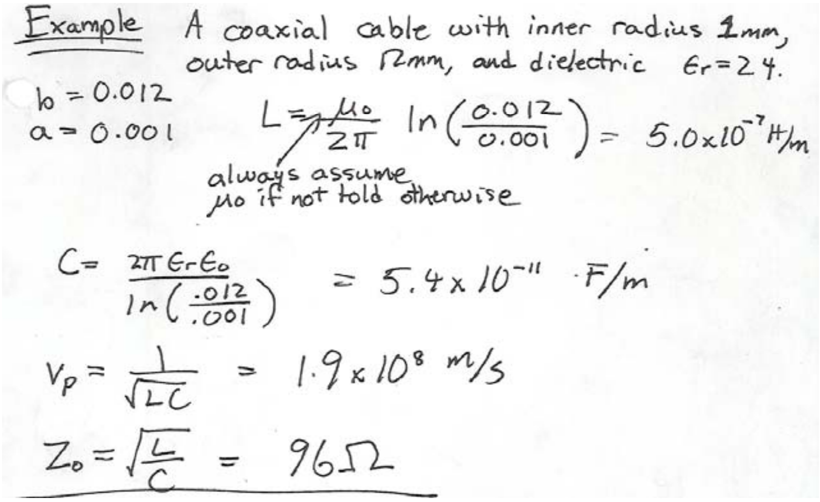


$$\epsilon_{eff} = \epsilon_o \left[\frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \frac{1}{\sqrt{1 + 12b/a}} \right]$$

$$V_p = \frac{1}{\sqrt{\mu\epsilon_{eff}}}$$

$$Z_o = \begin{cases} \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon_{eff}}} \ln\left(\frac{8b}{a} + \frac{a}{4b}\right) & a < b \\ \sqrt{\frac{\mu}{\epsilon_{eff}}} \frac{1}{\frac{a}{b} + 1.393 + 0.667\ln\left(\frac{a}{b} + 1.444\right)} & a > b \end{cases}$$

Example Calculation



Example A coaxial cable with inner radius 1mm, outer radius 12mm, and dielectric $\epsilon_r = 2.4$.

$b = 0.012$

$a = 0.001$

$$L = \frac{\mu_o}{2\pi} \ln\left(\frac{0.012}{0.001}\right) = 5.0\times10^{-7}\ H/m$$

always assume μ_o if not told otherwise

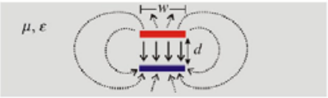
$$C = \frac{2\pi\epsilon_r\epsilon_o}{\ln\left(\frac{.012}{.001}\right)} = 5.4\times10^{-11}\ F/m$$

$$v_p = \frac{1}{\sqrt{LC}} = 1.9\times10^{8}\ m/s$$

$$Z_o = \sqrt{\frac{L}{C}} = 96\Omega$$

Summary of Transmission Line Topologies

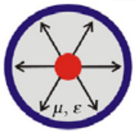
Parallel Plate



$$L = \mu\frac{d}{w} \qquad C = \varepsilon\frac{w}{d}$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} \qquad Z_0 = \frac{d}{w}\sqrt{\mu/\varepsilon}$$

Coaxial Cable



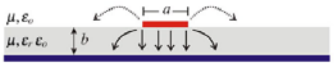
inner conductor radius, a

outer conductor radius, b

$$L = \frac{\mu}{2\pi}\ln(b/a) \qquad C = \frac{2\pi\varepsilon}{\ln(b/a)}$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon}} \qquad Z_0 = \frac{\ln(b/a)}{2\pi}\sqrt{\mu/\varepsilon}$$

Microstrip



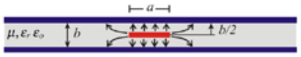
$$Z_0 = \begin{cases} \frac{1}{2\pi}\sqrt{\mu/\varepsilon_{eff}}\,\ln\left(\frac{8b}{a}+\frac{a}{4b}\right), & a < b \\ \sqrt{\mu/\varepsilon_{eff}}\,\frac{1}{a/b+1.393+0.667\ln(a/b+1.444)}, & a > b \end{cases}$$

$$\varepsilon_{eff} = 8.854\times10^{-12}\left[\frac{\varepsilon_r+1}{2}+\frac{\varepsilon_r-1}{2}\frac{1}{\sqrt{1+12b/a}}\right]\ \mathrm{F/m}$$

$$v_p = \frac{1}{\sqrt{\mu\varepsilon_{eff}}} \qquad L = \frac{Z_0}{v_p} \qquad C = \frac{1}{Z_0 v_p}$$

Other Transmission Line Topologies

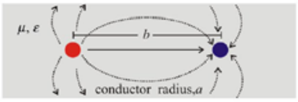
Stripline



$$a_{eff} = \begin{cases} a, & a > 0.35b \\ a - \left(0.35 - \frac{a}{b}\right)^2 b, & a < 0.35b \end{cases} \qquad L = \frac{Z_0}{v_p}$$

$$Z_0 = \frac{30\pi}{\sqrt{\varepsilon_r}} \frac{b}{a_{eff} + 0.441b} \qquad v_p = \frac{1}{\sqrt{\mu\varepsilon}} \qquad C = \frac{1}{Z_0 v_p}$$

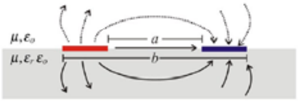
Two Wires / Twin Lead



$$L = \frac{\mu}{\pi} \cosh^{-1}\left(\frac{b}{2a}\right) \qquad C = \frac{\pi\varepsilon}{\cosh^{-1}\left(\frac{b}{2a}\right)}$$

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\varepsilon}} \cosh^{-1}\left(\frac{b}{2a}\right) \qquad v_p = \frac{1}{\sqrt{\mu\varepsilon}}$$

Coplanar Strip



$$Z_0 = \sqrt{\frac{\mu}{\varepsilon_{eff}}} \frac{\mathrm{K}(a/b)}{\mathrm{K}\left(\sqrt{1 - a^2/b^2}\right)} \qquad \varepsilon_{eff} = 8.854 \times 10^{-12} \left(\frac{\varepsilon_r + 1}{2}\right) \text{ F/m}$$

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} \qquad v_p = \frac{1}{\sqrt{\mu\varepsilon_{eff}}}$$

complete elliptic integral, 1st kind

Key:
+ 🟥 - 🟦 Conductor
▨ Dielectric
⟶ Confined Electric Field
⇢ Fringing Electric Field
$\mu = (4\pi \times 10^{-7})\,\mu_r$ (H/m)
$\varepsilon = (8.854 \times 10^{-12})\,\varepsilon_r$ (F/m)