Chapter 1

NOTES ON PHASORS

1.1 Time-Harmonic Physical Quantities

Time-harmonic analysis of physical systems is one of the most important skills for the electrical engineer to develop. Whether the application is power transmission, radio communications, data signaling, or laser emissions, the analysis of a physical system often requires physical quantities as either a single sinusoid or a superposition of multiple sinusoids. The most valuable analytical tool for studying sinusoidal physical quantities is the *phasor transform.*

1.1.1 Motivation for Phasors

There are many types of transforms in engineering and all of them have one thing in common: they are used to simplify physical calculations. The phasor transform is no different. It replaces a *time-harmonic* physical quantity with a single complex constant that can be manipulated more easily by the engineer.

A time-harmonic function is any physical quantity that is a sinusoidal function of time. Thus, a time-harmonic function, $f(t)$, has a general mathematical form given by the following cosine function:

$$f(t) = A\cos(2\pi ft + \phi) \tag{1.1.1}$$

where the three constants have the following meanings:

A: *amplitude* or *envelope* wave
f: harmonic *frequency*, with units of Hertz (Hz) or 1/s
ϕ: *phase* of the wave, with units of radians

The function $f(t)$ can represent any physical quantity – voltage, current, field, position, etc. – and is completely characterized by these three constants. Contrast this to a DC physical quantity which is characterized by a single constant.

Note: Radian Frequency

An alternative characterization to Equation (1.1.1) is often given by

$$f(t) = A\cos(\omega t + \phi)$$

where ω is *radian frequency* with units of radians-per-second. There are some disciplines of engineering that use ω instead of f in harmonic analysis. The engineering student should be able to move between these conventions easily by using the relationship $\omega = 2\pi f$.

1.1.2 Phasor Transform Definitions

The *phasor transform* is a one-on-one mapping of complex numbers (amplitudes and phases) to time-harmonic functions. In the official notation of transforms, we can denote the phasor transform as the operation $\mathcal{P}\{\}$:

$$\text{Forward Transform:} \quad \tilde{X} = \mathcal{P}\{x(t)\}$$

$$\text{Inverse Transform:} \quad x(t) = \mathcal{P}^{-1}\{\tilde{X}\}$$

We say that $\tilde{X}$ is in the *phasor domain* and $x(t)$ is in the *time domain*. The forward transform maps the function into the phasor domain and the inverse transform maps the phasor quantity back to the time domain. It is the convention of these notes to always mark a phasor quantity with a tilde $\tilde{()}$.

To calculate a phasor from a time-domain quantity, simply remove the cosine function and replace it with a complex exponential of the wave's phase offset. Mathematically, we write this as

$$x(t) = A\cos(2\pi f t + \phi) \;\longrightarrow\; \tilde{X} = A\exp(j\phi) \tag{1.1.2}$$

To take a phasor back into the time domain, use the following formula:

$$x(t) = \text{Real}\{\tilde{X}\exp(j2\pi f t)\} \tag{1.1.3}$$

An sample calculation of phasors is included in Example 1.1.

Example 1.1: Basic Phasor Transform

Problem: Convert the function $7\sin(2\pi t)$ into the phasor domain and then back into the time domain.

Solution:

1. To go into the phasor domain, we first recognize that if $x(t) = 7\sin(2\pi t)$, we may also write this as

$$x(t) = 7\cos\left(2\pi t - \frac{\pi}{2}\right)$$

Following the formula in Equation (1.1.2), we can write this function in the phasor domain as

$$\tilde{X} = 7\exp\left(-j\frac{\pi}{2}\right)$$

2. To go back into the time domain is straightforward:

$$\begin{aligned} x(t) &= \text{Real}\{7\exp\left(-j\frac{\pi}{2}\right)\exp(j2\pi ft)\} \\ &= \text{Real}\left\{7\exp\left(j\left[2\pi ft - \frac{\pi}{2}\right]\right)\right\} \\ &= \text{Real}\left\{7\cos\left(2\pi ft - \frac{\pi}{2}\right) + j7\sin\left(2\pi ft - \frac{\pi}{2}\right)\right\} \\ &= 7\cos\left(2\pi ft - \frac{\pi}{2}\right) \\ &= 7\sin\left(2\pi ft\right) \end{aligned}$$

For the last steps, we applied the Euler formula for complex exponentials: $\exp(jx) = \cos x + j\sin x$.

1.1.3 Rectangular and Polar Forms of Phasors

There are two shorthand ways of reporting phasor values. We have already used the convenient *rectangular* form, which takes the form of a complex number, $X + jY$. We call this the rectangular form because the pair (X, Y) can be envisioned as rectilinear coordinates on a Cartesian graph. If these xy-coordinates are converted to *polar* coordinates, with an amplitude R and an angle ϕ, then we may report the phasor in the short-hand power form: $R\angle\phi$.

The geometrical relationship between the rectangular and polar forms is summarized in Figure 1.1. Conversion between the two forms is straight-forward and is based on the Euler identity:

$$R\exp(j\phi) \quad = \quad = R\cos\phi + jR\sin\phi \quad = \quad X + jY \tag{1.1.4}$$

$$\text{Rectangular to Polar:} \quad R = \sqrt{X^2 + Y^2} \quad \phi = \begin{cases} \tan^{-1}\left(\frac{Y}{X}\right) & X > 0 \\ \pi + \tan^{-1}\left(\frac{Y}{X}\right) & X < 0 \end{cases}$$

$$\text{Polar to Rectangular:} \quad X = R\cos\phi \qquad Y = R\sin\phi \tag{1.1.5}$$

Each form has its advantages. For reporting phasor values, engineers usually use the polar form since the value R is essentially the amplitude of the oscillating sinusoid; the angle is best *presented* in degrees (eg. $7.2\angle 35°$) although the angle should be carried through *calculations* in radians.

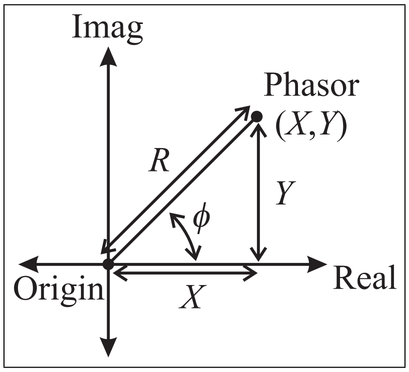


Figure 1.1. The rectangular form of a phasor marks a pair of Cartesian coordinates (X, Y) in the complex plane, with an alternate polar form representing magnitude R and phase ϕ.

Note: Pesky Inverse Tangent

Whenever doing a phasor or geometrical conversion, the inverse tangent formula in Equation (1.1.5) can frustrate even a seasoned engineer. There is an ambiguity in the basic $\tan^{-1}$ function which makes it only return values between $-\frac{pi}{2}$ and $\frac{pi}{2}$. Most computer programming languages provide assistance to engineers with two forms of the arctangent function: 1) a function `atan(z)` which calculates the basic inverse tangent of a and 2) a two-argument function `atan(y,x)` which includes the geometrical contingency of Equation (1.1.5) in the result.

The polar form is also most convenient for phasor calculations involving multiplication and division. Both operations have very simple polar-form expressions:

$$\begin{aligned} A\angle\phi_a \times B\angle\phi_b &= AB\angle(\phi_a + \phi_b) \\ \frac{A\angle\phi_a}{B\angle\phi_b} &= \frac{A}{B}\angle(\phi_a - \phi_b) \end{aligned} \tag{1.1.6}$$

The first step of any calculation involving complex multiplication or division is to convert any rectangular form phasors to polar form phasors.

Conversely, the rectangular form is most convenient for phasor calculations involving addition or subtraction. The first step of any calculation involving complex addition or subtraction is to convert any polar form phasors to rectangular form phasors.

1.2 Linear Time-Invariant Systems

Before we understand the utility of the phasor transform, we must first learn about *linear, time-invariant* (LTI) systems. This section defines LTI systems and illuminates some basic properties of sinusoids and phasors in these systems.

1.2.1 Definition of an LTI System

Most of the physical phenomena in basic engineering can be modeled as LTI systems. The first characteristic of an LTI system – linearity – implies that the system operates on the linear combination to two or more physical quantities in the same way that it operates on the individual quantities. Mathematically, if we represent the operation of a system on a function of time $f(t)$ as $\mathcal{H}\{f(t)\}$, then we may write that an LTI system must satisfy the following linearity relationship:

$$\mathcal{H}\{ax(t) + by(t)\} = a\mathcal{H}\{x(t)\} + b\mathcal{H}\{y(t)\} \tag{1.2.1}$$

where a and b are constants.

The second characteristic of an LTI system – time invariance – implies that the operation of a system on a signal is independent of absolute time. If an input, $x(t)$, to the system results in an output $y(t)$, then a delayed input of $x(t - t_0)$ will result always result in an output $y(t - t_0)$ for the LTI system. In other words, a shift in time of the input function only results in the same shift in time for the output function.

1.2.2 Output of an LTI System

The most common method to characterize an LTI system is with an *impulse response function*, $h(t)$. This function is defined to be the output of the system when a *Dirac impulse* is its input:

$$h(t) = \mathcal{H}\{\delta(t)\} \tag{1.2.2}$$

where $\delta(t)$ is the Dirac impulse function (defined and discussed in Appendix 1.A). This relationship is illustrated in Figure 1.2.

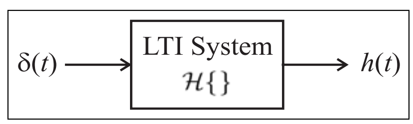


Figure 1.2. The impulse response $h(t)$ results when a linear, time-invariant system is excited at the input by an impulse function $\delta(t)$.

Let us see how to use the impulse response, $h(t)$, to calculate the output of an LTI system for an arbitrary input, $x(t)$. First, we note the following basic property of integrating a function with an impulse:

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t - \lambda)\, dt \tag{1.2.3}$$

This equation is called the *sifting integral* and is a basic property an impulse function. Thus, we may define the output, $y(t)$, of an LTI system with input, $x(t)$, as

$$\begin{aligned} y(t) &= \mathcal{H}\{x(t)\} \\ &= \mathcal{H}\{\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)\,dt\} \\ &= \int_{-\infty}^{\infty} x(\lambda)\underbrace{\mathcal{H}\{\delta(t-\lambda)\}}_{h(t-\lambda)}\,dt \end{aligned} \tag{1.2.4}$$

Note that, after substituting the sifting integral for $x(t)$, we used the linearity property of the system to bring the operator $\mathcal{H}\{\}$ inside the integral. After all, an integration is simply a summation of pieces. Since $x(\lambda)$ is not a function of time, we may treat it as a constant and move it outside the operator $\mathcal{H}\{\}$ as well.

We can make one more simplification to the last line of Equation (1.2.4) by invoking the property of time invariance. Recognize that the only time-varying component of this expression is just an impulse function shifted in time by λ. Thus, we can write the output as simply the impulse response function, $h(t)$, also shifted in time by λ. The final result is written as

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,dt \tag{1.2.5}$$

which takes the form of a *convolution integral*. The operation of convolution occurs so often in the physical sciences, that we often use the following short-hand notation:

$$y(t) = x(t) \otimes h(t) \tag{1.2.6}$$

Thus, the output of an LTI system may be calculated by *convolving* the input function, $x(t)$, with the system's impulse response, $h(t)$. Once $h(t)$ is known, the system output for any arbitrary input signal may be calculated by Equation (1.2.5).

1.2.3 Sine Wave In, Sine Wave Out

The operation of convolution has a very special property when one of the signals is a sine wave. Given the a sine wave with arbitrary amplitude, frequency, and phase, a convolution must have the following property:

$$A_1 \cos(2\pi ft + \phi_1) \otimes h(t) = A_2 \cos(2\pi ft + \phi_2)$$

This basic mathematical property has an important ramification for LTI systems: if the input of an LTI system is a sine wave of frequency f, then the output will also be a sine wave of frequency f. Only amplitudes and phases will change.

This is great news for analysis. Although it takes three pieces of information – amplitude, phase, and frequency – to characterize a wave, we only need to concern ourselves with amplitudes and phases in system analysis. Thus, complex phasors are so convenient for time-harmonic analysis. The two degrees of freedom in a phasor quantity (real and imaginary or, equivalently, magnitude and phase) track the amplitudes and phases of waves. The frequency can be taken for granted throughout the analysis.

Furthermore, phasors allow us to avoid many of the tedious convolution calculations for certain types of systems. If Equation (1.2.3) is true, then we can relate the phasor of the output waveform to the phasor of the input waveform through multiplication:

$$\tilde{Y} = \tilde{H}\,\tilde{X} \quad \text{or} \quad A_y \exp(j\phi_y) = A_h\, A_x \exp(j[\phi_h + \phi_x]) \tag{1.2.7}$$

Thus, an LTI system will change the amplitude of a sine wave by a factor of A_h and introduce a phase shift of $+\phi_h$ radians. The convolution integral becomes a simple multiplication in the phasor domain.

1.3 Summary

An outline of the notes on phasor transforms is given below:

- Phasors are complex numbers that are used to characterize the amplitudes and phases of time-harmonic physical quantities.
- An LTI system has the following properties:
  - The system is linear and time invariant.
  - The system is characterized by an impulse response.
  - A sinusoidal input will result in a sinusoidal output at the same frequency.
- Certain operations are much easier to perform in the phasor domain than in the time domain.
  - Superposition of waves becomes simple phasor addition.
  - Time-differentiation becomes multiplication by $2\pi f$.
  - Time-integration becomes division by $2\pi f$.
  - The effects of *any* LTI system can be represented with phasor multiplication.
- Phasor transforms may be used on vector functions, multivariable functions, and functions of 3D space.

1.A Singularity Functions

Singularity functions arise whenever descriptions of discontinuous or piecewise continuous functions are studied. The first singularity function we must define is the *unit step function* (also called the Heaviside unit step function):

$$\mathrm{u}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{1.A.1}$$

The unit step function is zero for all negative values of t and is one for all positive values of t.

The delta function is closely related to the unit step function. Often called the Dirac delta function or the impulse function, the delta function is the derivative of the unit step function.

$$\delta(t) = \frac{d\mathrm{u}(t)}{dt} \tag{1.A.2}$$

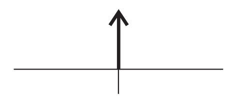
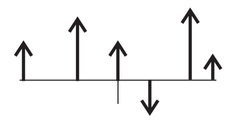
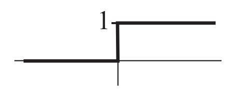
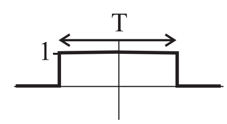
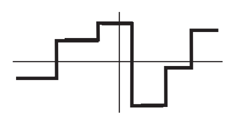
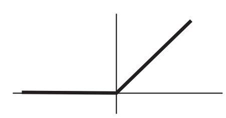
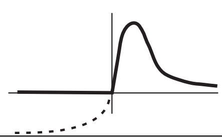
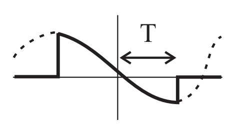
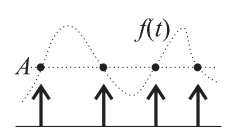
Since the unit step function has no slope for nonzero t, the delta function is 0 for all values $t \neq 0$. The discontinuity at $t = 0$ implies that the delta function has an infinite value at this point. The delta function represents an ideal pulse, a concentration of finite energy at a single point in time or other dependency. Table 1.1 records some of the most important properties of the delta function.

In graphs, the delta function is represented as a line segment terminating in an arrow; the length of the segment denotes amplitude. Delta functions and unit step functions may be combined in many ways to construct interesting signals and mathematical function shapes. Just a few examples of these combinations are shown in Table 1.2.

Table 1.1. Properties of the Delta Function, $\delta(t)$

Unit Step Relation	$\delta(t) = \frac{d\mathrm{u}(t)}{dt} \qquad \mathrm{u}(t) = \int_{-\infty}^{t} \delta(t_0)\, dt_0$
Pulse Definitions	$\delta(t) = \lim_{B\to\infty} B\mathrm{sn}(Bt) = \lim_{T\to 0} \frac{1}{T\sqrt{2\pi}} \exp\left(\frac{-t^2}{2T^2}\right)$
Reversal/Scaling	$\delta(-t) = \delta(t) \qquad \delta(at) = \frac{1}{\lvert a\rvert}\delta(t)$
Sifting Property	$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$
Sifting Integral	$f(t_0)\mathrm{u}(t_1-t_0) = \int_{-\infty}^{t_1} f(t)\delta(t-t_0)\, dt$
Integral Identity	$\delta(t) = \lim_{B\to\infty} \int_{-B/2}^{B/2} \exp(j2\pi ft)df \equiv \int_{-\infty}^{+\infty} \exp(j2\pi ft)df$
Function Argument	$\delta(y-f(t)) = \sum_i \left\lvert \frac{df(t_i)}{dt} \right\rvert^{-1} \delta(t-t_i)$ where t_i are the zeros of $f(t_i) = y$
Vector Notation	$\delta(\vec{\mathrm{r}}) = \delta(x)\delta(y)\delta(z)$ where $\vec{\mathrm{r}} = x\hat{\mathrm{x}} + y\hat{\mathrm{y}} + z\hat{\mathrm{z}}$
Spherical Expansion	$\delta(\vec{\mathrm{r}} - \vec{\mathrm{r}}_0) = \frac{\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)}{r_0^2 \cos(\varphi_0)}$

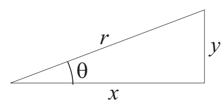
Table 1.2. Common Examples of Singularity Functions

Function Type	Mathematical Form	Function Shape
Delta Function	$\delta(t)$	
Impulse Train	$\sum_i A_i \delta(t - t_i)$	
Unit Step Function	$\mathrm{u}(t)$	
Box Function	$\mathrm{u}(\frac{T}{2} - \lvert t \rvert)$	
Staircase Function	$\sum_i A_i \mathrm{u}(t - t_i)$	
Ramp Function	$t\mathrm{u}(t)$	
Clipped Function	$f(t)\mathrm{u}(t)$	
Framed Function	$f(t)\mathrm{u}(T - \lvert t \rvert)$	
Crossing Counter	$\left\lvert \frac{df(t)}{dt} \right\rvert \delta(f(t) - A)$	

1.B Trigonometric Relationships

Table 1.3 has been included as a general summary of useful trigonometric relations.

Table 1.3. Useful Trigonometric Identities

Geometrical Definitions $\cos(\theta) = \frac{x}{r}$ $\sin(\theta) = \frac{y}{r}$ $\tan(\theta) = \frac{y}{x}$  $\cos^2(\theta) + \sin^2(\theta) = 1$	**Complex Exponents** $\exp(j\theta) = \cos(\theta) + j\sin(\theta)$ $\cos(\theta) = \frac{\exp(j\theta)+\exp(-j\theta)}{2}$ $\sin(\theta) = \frac{\exp(j\theta)-\exp(-j\theta)}{j2}$
Double-Angle Formulas $\cos(2\theta) = \begin{cases} 2\cos^2(\theta) - 1 \text{ or} \\ 1 - 2\sin^2(\theta) \text{ or} \\ \cos^2(\theta) - \sin^2(\theta) \end{cases}$ $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$	**Series Expansion** $\exp(j\theta) = \sum_{n=0}^{\infty} \frac{(j\theta)^n}{n!}$ $\cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}$ $\sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$
Addition Formulas $\cos(\theta_1+\theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ $\sin(\theta_1+\theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)$	**Half-Angle Formulas** $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos(\theta)}{2}}$ $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos(\theta)}{2}}$
Inverse Derivatives $\frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$ for $\|x\| \leq 1$ $\frac{d\cos^{-1}(x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$ for $\|x\| \leq 1$ $\frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$	**Derivatives** $\frac{d\sin(\theta)}{d\theta} = \cos(\theta)$ $\frac{d\cos(\theta)}{d\theta} = -\sin(\theta)$ $\frac{d\tan(\theta)}{d\theta} = 1 + \tan^2(\theta)$