Chapter 2

NOTES ON WAVE EQUATIONS

2.1 Maxwell's Equations

Wave equations describe a particular type of phenomenon exhibited by Maxwell's equations under sinusoidal excitation. In this section, we discuss different forms of Maxwell's equations before delving into the actual wave equations.

2.1.1 Full Maxwell's Equations

The full set of Maxwell's equations come in two forms. The first form is called either the *integral* or *volume* form and is shown in Table 2.1. These set of four equations show the interrelationships of the basic physical quantities studied in electromagnetics, which are summarized in Table 2.2. This form of Maxwell's equations elegantly relates volume- and area-integrals of different field components.

Table 2.1. Maxwell's Equations (Integral Form)

$\oint_L \vec{\mathsf{H}} \cdot d\vec{l} = \iint_A \left[\vec{\mathsf{J}} + \frac{\partial \vec{\mathsf{D}}}{\partial t}\right] \cdot d\hat{n}$	$\oiint_A \vec{\mathsf{B}} \cdot d\hat{n} = 0$
$\oint_L \vec{\mathsf{E}} \cdot d\vec{L} = -\iint_A \frac{\partial \vec{\mathsf{B}}}{\partial t} \cdot d\hat{n}$	$\oiint_A \vec{\mathsf{D}} \cdot d\hat{n} = \iiint_V \mathsf{q}\, dV$

The notation of the integral form of Maxwell's equations, while very elegant, is somewhat cumbersome. The *differential* or *point* form of Maxwell's equations is a more compact and easier set of equations to manipulate. Table 2.3 summarizes the four point-form equations of Maxwell, which are mathematically *identical* to the relationships in Table 2.1 – despite the fact that they look nothing alike.

Table 2.2. Quantities and units in Maxwell's equations.

Variable	Units	Technical Name
$\vec{\mathsf{E}}$	Volts/m	Electric Field
$\vec{\mathsf{H}}$	Amps/m	Magnetic Field
$\vec{\mathsf{D}}$	Coulombs/m^2	Electric Flux Density
$\vec{\mathsf{B}}$	Webers/m^2	Magnetic Flux Density
$\vec{\mathsf{J}}$	Amps/m^2	Current Density
q	Coulombs/m^3	Charge Density (Volume)

Table 2.3. Maxwell's Equations (Point Form)

$$\nabla \times \vec{\mathsf{H}}(\vec{r},t) = \frac{\partial \vec{\mathsf{D}}(\vec{r},t)}{\partial t} + \vec{\mathsf{J}}(\vec{r},t) \qquad \nabla \cdot \vec{\mathsf{B}}(\vec{r},t) = 0$$

$$\nabla \times \vec{\mathsf{E}}(\vec{r},t) = -\frac{\partial \vec{\mathsf{B}}(\vec{r},t)}{\partial t} \qquad \nabla \cdot \vec{\mathsf{D}}(\vec{r},t) = \mathsf{q}(\vec{r},t)$$

Note: Oliver Heaviside (British Scientist, 1850-1925)

Oliver Heaviside lived in Victorian England and, at the turn of the nineteenth century outlined some of the most important principles of electromagnetics. His series of *Electromagnetic Theory* textbooks introduced the use of divergence ($\nabla\cdot$) and curl ($\nabla\times$) operators. Before Heaviside, electromagnetic scientists simply wrote the differential form of Maxwell's equations in long-hand notation. Heaviside made a number of electromagnetic discoveries, including the correct prediction that the Earth's conducting ionosphere would trap outgoing radio waves and bend them back to the ground, thereby allowing long-distance wireless communications.

2.1.2 Phasor Form Maxwell's Equations

Since we are often interested in electromagnetic waves, it helps to simplify Maxwell's equations for the special case of *time-harmonic* fields. These fields, whose only time dependence is sinusoidal, may be represented in the phasor domain just like scalar time-harmonic functions. For example, we may write the relationship between time-harmonic electric field, $\vec{\mathsf{E}}(\vec{r},t)$, and its phasor field, $\tilde{\vec{\mathsf{E}}}(\vec{r})$, as

$$\vec{\mathsf{E}}(\vec{r},t) = \mathrm{Real}\{\tilde{\vec{\mathsf{E}}}(\vec{r})\exp(j2\pi ft)\}$$

where f is the frequency of radiation. The phasor representation simplifies the analysis of electromagnetic waves since it completely removes the time dependency from all field components.

Thus, we may write all of the point-form Maxwell's equations in phasor form as shown in Table 2.4. We take advantage of the fact that time-differentiation becomes a simple $j2\pi f$-multiplication in the phasor domain. Since all of the vector differential operators are with respect to space and not time, they remain unchanged in the phasor form of Maxwell's equations.

Table 2.4. Maxwell's Equations (Phasor Form)

$\nabla \times \tilde{\vec{\mathsf{H}}}(\vec{\mathrm{r}}) = j2\pi f \tilde{\vec{\mathsf{D}}}(\vec{\mathrm{r}}) + \tilde{\vec{\mathsf{J}}}(\vec{\mathrm{r}})$	$\nabla \cdot \tilde{\vec{\mathsf{B}}}(\vec{\mathrm{r}}) = 0$
$\nabla \times \tilde{\vec{\mathsf{E}}}(\vec{\mathrm{r}}) = -j2\pi f \tilde{\vec{\mathsf{B}}}(\vec{\mathrm{r}})$	$\nabla \cdot \tilde{\vec{\mathsf{D}}}(\vec{\mathrm{r}}) = \tilde{\mathsf{q}}(\vec{\mathrm{r}})$

2.2 Maxwell's Equations in Simple Media

The space and material through which a wave travels constitute the *medium* of propagation. There are countless types of propagation media, but we are primarily concerned with *simple media*. A simple medium is said to be linear, isotropic, homogeneous, and sourceless. We will discuss what each of these terms means individually.

2.2.1 Linearity and Isotropy

A medium is said to be *linear* if all flux density vectors are proportional to their corresponding field components. Electric flux density, $\vec{\mathsf{D}}$, and electric field, $\vec{\mathsf{E}}$, must be proportional to one another in a linear medium. Likewise, magnetic flux density, $\vec{\mathsf{B}}$, and magnetic field, $\vec{\mathsf{H}}$, must also be proportional to one another.

A medium is said to be *isotropic* if its material properties do not vary as a function of field orientation. A medium is said to be *anisotropic* (not isotropic) if the relationship between flux densities and corresponding fields change as a function of field direction. The most common form of anisotropy that we encounter in real life are polarizing sunglasses; the vertical orientation of electric field experiences different material properties than the horizontal orientation of electric field.

Taken together, the attributes of linearity and isotropy signify a material that can be characterized by two scalar constants: permittivity (ϵ) and permeability (μ). Mathematically, we write this condition as

$$\text{Linear, Isotropic Medium:} \quad \vec{\mathsf{D}} = \epsilon\vec{\mathsf{E}}, \quad \vec{\mathsf{B}} = \mu\vec{\mathsf{H}} \tag{2.2.1}$$

Note that it is possible to have a linear, anisotropic medium. For such a medium, we cannot relate flux density and field vectors by scalar constants. Instead, we must use matrix-valued ϵ and μ. Due to their nasty mathematical complexity, anisotropic media are rarely studied in undergraduate electromagnetics.

2.2.2 Homogeneity and Source-Free Space

A medium is said to be *homogeneous* if its material properties do not depend on space. For the linear, isotropic medium, this implies that ϵ and μ are not functions of space. Homogeneous media are highly idealized, since they must consist of the same material in all parts of space. As soon as a differing "chunk" of material is inserted into the material, the medium becomes inhomogeneous. Therefore, we usually invoke the condition of homogeneity to represent finite regions of space where objects and scatterers are absent.

A medium is said to be *source-free* if it contains no electromagnetic charges or currents. Mathematically, we write this condition as

$$\text{Sourceless Medium:} \quad \vec{\mathsf{J}} = \vec{0}, \quad \mathsf{q} = 0 \tag{2.2.2}$$

We must have zero-valued current density, $\vec{\mathsf{J}}$, and electric charge, q in a sourceless medium.

2.2.3 Simple Medium Form of Maxwell's Equations

When all four conditions of a simple medium are considered – linearity, isotropy, homogeneity, and source-free space – they lead to a much simplified form of Maxwell's equations. Table 2.5 summarize the simple media Maxwell's equations, as derived for time-harmonic fields. Note that there are now only two field quantities, $\tilde{\vec{\mathsf{E}}}$ and $\tilde{\vec{\mathsf{H}}}$, to solve. Now we are ready to develop the wave equations.

Table 2.5. Maxwell's Equations (Simple Medium)

$\nabla \times \tilde{\vec{\mathsf{H}}}(\vec{r}) = j2\pi f \epsilon \tilde{\vec{\mathsf{E}}}(\vec{r})$	$\nabla \cdot \tilde{\vec{\mathsf{H}}}(\vec{r}) = 0$
$\nabla \times \tilde{\vec{\mathsf{E}}}(\vec{r}) = -j2\pi f \mu \tilde{\vec{\mathsf{H}}}(\vec{r})$	$\nabla \cdot \tilde{\vec{\mathsf{E}}}(\vec{r}) = 0$

2.3 The Wave Equations

One of the most interesting phenomena to result from Maxwell's equations is *wave propagation*. This section develops the basic wave equations used in radio and optics to describe how waves propagate in space.

2.3.1 The Vector Wave Equation

Through a series of manipulations (outlined in Table 2.6), we can derive the *vector wave equation* from the phasor form of Maxwell's equations in a simple medium. The resulting vector wave equation is given by

$$(\nabla \times \nabla \times -k^2) \begin{pmatrix} \tilde{\vec{\mathsf{H}}} \\ \tilde{\vec{\mathsf{E}}} \end{pmatrix} = \mathbf{0} \tag{2.3.1}$$

where k is the *wavenumber* of radiation:

$$k = 2\pi f \sqrt{\epsilon\mu} = \frac{2\pi}{\lambda} \tag{2.3.2}$$

The vector wave equation can be thought of as one big differential operator: a double-curl operation with a constant subtracted from it. This operates on both $\tilde{\vec{\mathsf{E}}}$ and $\tilde{\vec{\mathsf{H}}}$ and, for all time-harmonic fields in a simple medium, must 0 for all vector components. Studying the vector wave equation is usually beyond the scope of introductory electromagnetic, but there is one more simplification that can be made to the vector wave equation.

2.3.2 The Scalar Wave Equation

Now let us derive a simplified version of the vector wave equation. First, we will apply the following vector calculus identity:

$$\nabla \times \nabla \times \vec{\mathrm{A}} = \nabla(\nabla \cdot \vec{\mathrm{A}}) - \nabla^2 \vec{\mathrm{A}} \tag{2.3.3}$$

This identity states that the double-curl operation on a vector may be rewritten as a gradient-divergence *and* a Laplacian operation. However, we will be applying this relationship to fields in a simple medium, which means that the divergence of $\tilde{\vec{\mathsf{H}}}$ and $\tilde{\vec{\mathsf{E}}}$ are going to evaluate to 0:

$$\nabla \cdot \begin{pmatrix} \tilde{\vec{\mathsf{H}}} \\ \tilde{\vec{\mathsf{E}}} \end{pmatrix} = \mathbf{0} \tag{2.3.4}$$

Thus, we may rewrite Equation (2.3.1) as the following scalar wave equation:

$$(\nabla^2 + k^2) \begin{pmatrix} \tilde{\vec{\mathsf{H}}} \\ \tilde{\vec{\mathsf{E}}} \end{pmatrix} = \mathbf{0} \tag{2.3.5}$$

Equation (2.3.5) is also referred to as the *Helmholtz* wave equation.

Table 2.6. How to derive the vector wave equation for $\tilde{\vec{\mathsf{H}}}$ and $\tilde{\vec{\mathsf{E}}}$.

Step	$\vec{\mathsf{H}}$-field	$\vec{\mathsf{E}}$-field
1) Start with a Maxwell curl equation in a simple medium.	$\nabla \times \tilde{\vec{\mathsf{H}}}(\vec{r}) = j2\pi f \epsilon \tilde{\vec{\mathsf{E}}}(\vec{r})$	$\nabla \times \tilde{\vec{\mathsf{E}}}(\vec{r}) = -j2\pi f \mu \tilde{\vec{\mathsf{H}}}(\vec{r})$
2) Take the curl of each side.	$\nabla \times \nabla \times \tilde{\vec{\mathsf{H}}}(\vec{r}) = j2\pi f \epsilon \left[\nabla \times \tilde{\vec{\mathsf{E}}}(\vec{r}) \right]$	$\nabla \times \nabla \times \tilde{\vec{\mathsf{E}}}(\vec{r}) = -j2\pi f \mu \left[\nabla \times \tilde{\vec{\mathsf{H}}}(\vec{r}) \right]$
3) Substitute the alternate Maxwell curl equation on the right-hand side.	$\nabla \times \nabla \times \tilde{\vec{\mathsf{H}}}(\vec{r}) = j2\pi f \epsilon \left[-j2\pi f \mu \tilde{\vec{\mathsf{H}}}(\vec{r}) \right]$	$\nabla \times \nabla \times \tilde{\vec{\mathsf{E}}}(\vec{r}) = -j2\pi f \mu \left[j2\pi f \epsilon \tilde{\vec{\mathsf{E}}}(\vec{r}) \right]$
4) Simplify and group terms on the left-hand side.	$(\nabla \times \nabla \times - \underbrace{4\pi^2 f^2 \epsilon \mu}_{k^2}) \tilde{\vec{\mathsf{H}}}(\vec{r}) = \vec{0}$	$(\nabla \times \nabla \times - \underbrace{4\pi^2 f^2 \epsilon \mu}_{k^2}) \tilde{\vec{\mathsf{E}}}(\vec{r}) = \vec{0}$

Note: Laplacian Operator, ∇^2

The ∇^2 is called the *Laplacian* operator and takes the following form:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

This operator is commonly used in electrostatics to operate on voltages. The Laplacian operates on a scalar quantity and returns a scalar quantity. In Equation (2.3.5), it appears to be operating on a vector, but this is actually a shorthand notation for the Laplacian of each individual field component:

$$\nabla^2 \tilde{\vec{\mathsf{E}}} = \nabla^2 E_x \hat{x} + \nabla^2 E_y \hat{y} + \nabla^2 E_z \hat{z}$$

2.3.3 Dissecting The Wave Equation

Equation (2.3.5) is much simpler to solve than the vector equation, although at first glance one wonders why it is called a *scalar* wave equation since the $(\nabla^2 + k^2)$ still appears to be operating on the electric and magnetic field vectors. However, we refer to this equation as scalar because the operation on all of the field components is *separable*. Thus, we can write the entire system of Helmholtz wave equations as 6 separable scalar equations:

$$(\nabla^2 + k^2) \begin{pmatrix} \tilde{H}_x \\ \tilde{H}_y \\ \tilde{H}_z \\ \tilde{E}_x \\ \tilde{E}_y \\ \tilde{E}_z \end{pmatrix} = \mathbf{0} \tag{2.3.6}$$

This separability makes the solution of the Helmholtz equations much easier than the vector wave equation. This is entirely a result of the simple medium that we assumed in deriving the wave equations.

There is an interesting parallel between free wave propagation and transmission line propagation. As a starting point, let us look at the wave equation for the single x-component of magnetic field:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \tilde{H}_x = 0 \tag{2.3.7}$$

This takes a strikingly similar form to the phasor-form differential equation for transmission line propagation:

$$\left(\frac{\partial^2}{\partial z^2} + \beta^2 \right) \tilde{H}_x = 0 \tag{2.3.8}$$

where β is also wavenumber. The only substantial difference is that Equation (2.3.7) is a function of three-dimensional space, whereas Equation (2.3.8) is only a function of one-dimensional position. But both equations define lossless, constant-velocity propagation of electromagnetic waves. Thus, many of the concepts studied in transmission line theory will help in understanding unbounded wave propagation.