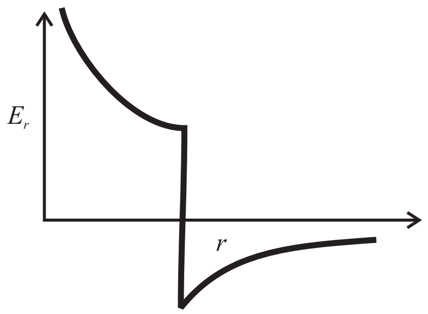
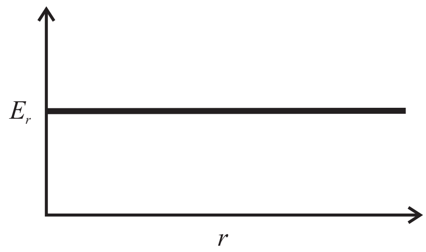
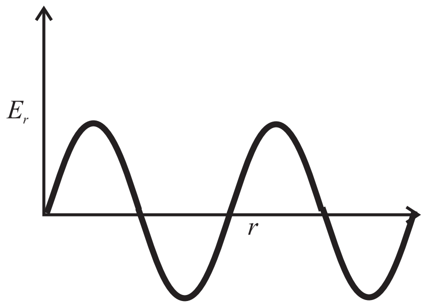
ECE 3025: Electromagnetics
Solutions to TEST 2 (Spring 2012)

1. **Numerical Evaluation of Laplace's Equation:** The center of the diagram (the tip of the lightning rod) experiences the strongest field (verified by the steepest electrostatic potential difference between cells). The missing boxes should be the average of the four adjacent squares (it is wrong to include the diagonals!) for a total of 405/4 V or 101.25 V.

2. **Charge Distributions:**







3. **Voltage of a Tube Charge:** The field solutions follows in a straight-forward manner from integration of the constant charge ρ_v over the tubular volume for an observation point on the z-axis, (0,0,z):

$$
\begin{aligned}
\vec{E}(0,0,z) &= \int_{\mathcal{V}} \frac{\rho_v\,(\vec{r}-\vec{r}')\,dv'}{4\pi\epsilon\|\vec{r}-\vec{r}'\|^3} \\
&= \frac{\rho_v}{4\pi\epsilon}\int_a^b d\rho' \int_{-D/2}^{D/2} dz' \int_0^{2\pi} \rho'\,d\phi' \frac{\overbrace{(0-x')\hat{x}+(0-y')\hat{y}}^{\text{0 by symmetry}}+(z-z')\hat{z}}{\|(0-x')\hat{x}+(0-y')\hat{y}+(z-z')\hat{z}\|^3} \\
&= \frac{\rho_v\hat{z}}{4\pi\epsilon}\int_a^b d\rho' \int_{-D/2}^{D/2} dz' \int_0^{2\pi} \rho'\,d\phi' \frac{z-z'}{\left(\underbrace{x'^2+y'^2}_{\rho'^2}+(z-z')^2\right)^{\frac{3}{2}}}
\end{aligned}
$$

$$= \frac{\rho_v z\hat{z}}{2\epsilon} \int_a^b d\rho' \int_{-D/2}^{D/2} dz' \frac{\rho'}{(\rho'^2 + (z-z')^2)^{\frac{3}{2}}}$$

Note that the z' term in the numerator was dropped because of its odd symmetry in the integral; it will eventually integrate to zero. Anyone who got this far in the problem received full credit. I would have been impressed beyond all reason if the student continued to produce

$$\vec{E}(0,0,z) = \frac{\rho_v z\hat{z}}{2\epsilon} \ln\left(\left[\frac{\sqrt{b^2 + (z+D/2)^2} + z + D/2}{\sqrt{b^2 + (z-D/2)^2} + z - D/2}\right]\left[\frac{\sqrt{a^2 + (z+D/2)^2} + z + D/2}{\sqrt{a^2 + (z-D/2)^2} + z - D/2}\right]\right)$$

The voltage solution follows similarly:

$$\begin{aligned} V(0,0,z) &= \int_{\mathcal{V}} \frac{\rho_v \, dv'}{4\pi\epsilon\|\vec{r} - \vec{r'}\|} \\ &= \frac{\rho_v}{4\pi\epsilon} \int_a^b d\rho' \int_{-D/2}^{D/2} dz' \int_0^{2\pi} \rho' \, d\phi' \frac{1}{\|(0-x')\hat{x} + (0-y')\hat{y} + (z-z')\hat{z}\|} \\ &= \frac{\rho_v}{2\epsilon} \int_a^b d\rho' \int_{-D/2}^{D/2} dz' \frac{\rho'}{\sqrt{\rho'^2 + (z-z')^2}} \end{aligned}$$

Anyone who got this far in the problem received full credit. The analytical solution to this last double integral is too hideous to write in this space.