ECE 3025: Solution to Homework 1

Introduction to Time-Domain Transmission Lines

Prof. Durgin, Spring 2011, TTh 1:35pm

Assigned Problems

1. Peterson & Durgin, 2.5 (10 points).

2.5 The following represent solutions of the transmission line differential equations:

$$v(z,t) = 60\cos(1000t - 5z) - 120\pi f\left(2t + \frac{z}{100} + 12\right) \text{ V}$$

$$i(z,t) = 2\pi f\left(2t + \frac{z}{100} + 12\right) + \cos(1000t - 5z) \text{ A}$$

for some arbitrary function $f(u)$. Answer the following questions about this solution:

(a) What is the forward propagating voltage waveform?

$$v^{+}(z,t) = 60\cos(1000t - 5z)$$

(b) What is the backward-propagating current waveform?

$$i^{-}(z,t) = 2\pi f\left(2t + \frac{z}{100} + 12\right)$$

(c) What is the characteristic impedance of this line?

$$Z_0 = \frac{v^{+}(z,t)}{i^{+}(z,t)} = \frac{v^{-}(z,t)}{i^{-}(z,t)} = 60\ \Omega$$

(d) What is the velocity of propagation for a signal on this line?

$$v_p = \frac{1000}{5} = 2(100) = 200$$

(e) What is the total voltage at $t = 0$ at the input end of the line ($z = 0$)?

$$v(0,0) = 60 - 120\pi f(12) \text{ V}$$

2. Peterson & Durgin, 2.6 (10 points).

2.6 A bankrupt student discovers a 15 m length of coaxial cable in the lab with connectors on both ends, so it is not possible to see the center conductor. Another student bets her $50 that she cannot determine the outside diameter of the inner conductor or the nature of the insulating material between the conductors without cutting the cable open.

The student takes the bet. First, she holds the cable up to a magnet and determines that since there is no attraction, the cable is constructed only of non-magnetic materials with $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m. Second, she uses a dual trace oscilloscope to measure the time it takes a pulse to travel down the line, and determines that it takes 70 nanoseconds. Third, she uses the capacitance meter on her digital voltmeter to determine the total capacitance of the 15-meter cable to be 3 nF. Finally, by measuring the outer diameter of the cable and guessing at the thickness of the outer conductor, she estimates that the outer conductor has an inside radius of $b = 6$ mm. With this information, she is able to answer the following questions and win the bet! Now it is your turn:

(a) What is the propagation velocity on the transmission line?

$$v_p = \frac{15\,\text{m}}{70 \times 10^{-9}\,\text{s}} = 2.143 \times 10^8 \text{ m/s}$$

(b) What is the relative permittivity of the insulating material between the conductors of the coax?

Use the fact that the propagation velocity can be written as

$$2.143 \times 10^8 \text{ m/s} = \frac{2.998 \times 10^8}{\sqrt{\varepsilon_r}} \quad \text{to obtain:} \quad \varepsilon_r = 1.96$$

(c) What is the capacitance per unit length for the cable?

$$C = \frac{3 \times 10^{-9}\,\text{F}}{15\,\text{m}} = 2.000 \times 10^{-10} \text{ F/m}$$

(d) What is the diameter of the center conductor?

From part (e), use $Z_0 = 23.3\ \Omega = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}} \frac{\ln(b/a)}{2\pi}$

to solve for $a = 3.478$ mm

(e) What is the characteristic impedance of the line?

Use $L = \frac{\mu_0 \varepsilon_0 \varepsilon_r}{C} = 1.09 \times 10^{-7}$ H/m

to obtain $Z_0 = \sqrt{\frac{L}{C}} = 23.3\ \Omega$

3. Peterson & Durgin, 2.7 (10 points).

2.7 Design a coaxial cable with characteristic impedance $Z_0 = 50\ \Omega$. The cable must have an outer conductor radius of 1 cm and a solid inner copper core of radius a. You must select a dielectric with relative permittivity ε_r and a value for the conductor core radius a that achieves the desired impedance while minimizing the cost-per-meter of the cable. Assume that copper conductor costs \$2000 per cubic meter and the dielectric material cost is given by the function $\$(200 + 25\varepsilon_r)$ per cubic meter. (Note that higher permittivity dielectrics are more expensive!) In addition to ε_r and a, determine the cost-per-foot of your optimum design as well as the velocity of propagation, the inductance per unit length, and the capacitance per unit length.

Hint: It is probably easiest to plot the total cost vs. ε_r and find the optimal design parameters by a visual inspection of the graph.

First, from geometry we know that the volumes of copper and dielectric (per unit length of cable) are given by the following expressions:

$$\text{Cu vol/m: } a^2\pi \qquad \text{Dielectric vol/m: } (b^2 - a^2)\pi$$

The value b is fixed at 0.01m. The per-unit cost of each medium is given by:

$$\text{Cu: } 2000a^2\pi \text{ \$/m} \qquad \text{Dielectric: } (200 + 25\epsilon_r)(b^2 - a^2)\pi \text{ \$/m}$$

(Note: there would also be some outer conductor coating, but this is a fixed cost since the cable radius is static.)

We also know that a will be a function of the dielectric permittivity ϵ_r. We must invert the coaxial equation from the notes:

$$Z_0 = \frac{\ln\left(\frac{b}{a}\right)}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \quad \longrightarrow \quad a = b\exp\left(-2\pi Z_0\sqrt{\frac{\epsilon_r \epsilon_0}{\mu_0}}\right)$$

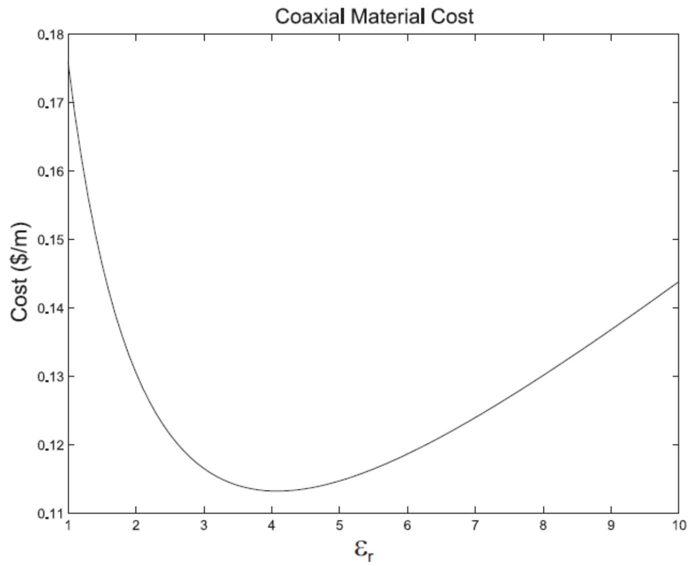
This expression allows us to calculate the inner conductor radius as a function of permittivity (unknown) and target impedance (known). Thus, to compute the total cost of the dielectric and copper core, we can construct the following expressions through substitution:

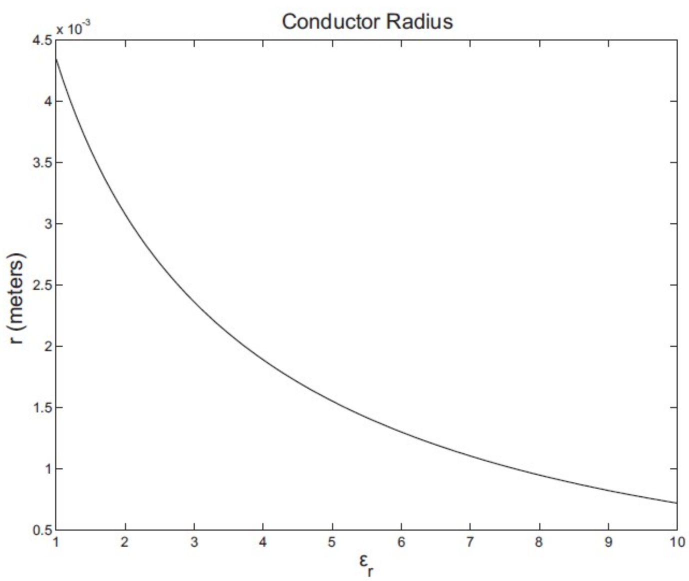
$$\text{Cu Cost: } 2000\pi\left[0.01\exp\left(-2\pi(50)\sqrt{\frac{\epsilon_r\epsilon_0}{\mu_0}}\right)\right]^2$$

$$\text{Dl Cost: } (200 + 25\epsilon_r)\pi\left((0.01)^2 - \left[0.01\exp\left(-2\pi(50)\sqrt{\frac{\epsilon_r\epsilon_0}{\mu_0}}\right)\right]^2\right)$$

The total cost of the dielectric and copper core is the sum of these two values.

Now let's plot total cost as a function of dielectric parameter ϵ_r along with the inner core radius:





From visual inspection, we see that the cheapest 50 Ω cable can be made with $\epsilon_r \approx 4.0$ and $a \approx 1.9$ mm at a total cost of 11.3 cents/m. The velocity of propagation would be 1.5×10^8 m/s ($v_p = \frac{1}{\sqrt{\epsilon_r \epsilon_0 \mu_0}}$). The per-unit capacitance and inductance are 1.3×10^{-10} F/m and 3.3×10^{-7} H/m, respectively.

The Matlab code used to make the graphs for this solution is listed below:

```
b = 0.01;                 % size (in meters) of outer radius
er = 1:.1:10;             % range of relative permittivities
e0 = 8.85e-12;            % free space permittivity
u0 = 4*pi*10^-7;          % free space permeability
Z0 = 50;                  % target impedance of the transmission line)
alpha = 2000;             % cost of conductor per cubic meter
beta = 200 + 25*er;       % cost of dielectric per cubic meter

a = b*exp(-2*pi*Z0*(e0/u0)^.5*(er.^.5));
Cost = pi*(alpha * a.^2 + (b^2-a.^2).*beta);
figure(1); plot( er, a );
xlabel('\epsilon_r'); ylabel('r (meters)');
title('Conductor Radius');

figure(2); plot( er, Cost );
xlabel('\epsilon_r'); ylabel('Cost ($/m)');
title('Coaxial Material Cost');
```

4. Peterson & Durgin, 2.8 (10 points).

2.8 Design a microstrip transmission line with $Z_0 = 100\ \Omega$ that will be etched onto a dielectric substrate with $\varepsilon_r = 3$ and a thickness of 4 mm. What should be the width of the microstrip line?

The easiest way to calculate the width of the microstrip is to use the inversion formula in the notes:

$$a = b\left[\frac{8\exp(A)}{\exp(2A) - 2}\right]$$

where

$$A = \frac{Z_0}{60}\sqrt{\frac{\epsilon_r + 1}{2}} + \frac{\epsilon_r - 1}{\epsilon_r + 1}\left(0.23 + \frac{0.11}{\epsilon_r}\right)$$

$$B = \frac{377\pi}{2Z_0\sqrt{\epsilon_r}}$$

When these expressions are evaluated for the parameters discussed in the problem statement, the results are $A = 2.49$, $B = 3.42$, and $a = 2.7$ mm.

5. Peterson & Durgin, 2.9 (10 points).

2.9 The following represent invalid solutions of the transmission line differential equations. Explain why each is impossible for a linear transmission line in terms of basic physical properties (i.e., characteristic impedance, pulse shape, attenuation, velocity of propagation, etc.) No credit will be given for purely mathematical answers.

(a)
$$v(z,t) = 100\cos(2\pi ft - z) + 50\sin(2\pi ft + z)$$
$$i(z,t) = 5\cos(2\pi ft - z) - 5\sin(2\pi ft + z)$$

(b)
$$v(z,t) = 20\frac{\sin(t-5z)}{t-5z}$$
$$i(z,t) = \sin(t-5z)$$

(c)
$$v(z,t) = 75\exp\left(-|t-z|^2\right) + 75\exp\left(-\left|t+\frac{z}{2}\right|^2\right)$$
$$i(z,t) = \exp\left(-|t-z|^2\right) - \exp\left(-\left|t+\frac{z}{2}\right|^2\right)$$

1. The ratio of voltage-to-current for the forward propagating wave is 20Ω; for the backwards propagating wave, this ratio is 10Ω. A lossless, linear transmission line has the same impedance for waveforms in both directions.

2. The current waveform has a different functional shape that the voltage waveform.

3. The backwards-propagating wave experiences exponential *growth* as it travels. For a passive device, this is not possible.

4. The forwards- and backwards-propagating waveforms are traveling at different velocities on this line.