ECE 3025 Homework 10: Magnetostatics

Solutions

1. Part A: $I = \pi R^2 J_0$ Part B: $I = L^2 J_0$

2. Square-loop current:

   (a) Pick an integral based on the charge/current distribution:

$$\vec{H}(\vec{r}) = \oint_L \frac{Id\hat{l} \times (\vec{r} - \vec{r}')}{4\pi|\vec{r} - \vec{r}'|^3}$$

   (b) Expand variables of integration and point of observation:

$$\text{Variable of Integration: } \vec{r}' = x'\hat{x} + y'\hat{y}$$

$$\text{Point of Observation: } \vec{r} = z\hat{z}$$

   Note that for this problem, all of our integration occurs in the xy-plane and all of our observation points are limited to the z-axis.

   (c) Pick a differential element of integration:

$$\text{Current in } x\text{-direction: } d\hat{l} = dx'\hat{x}$$

$$\text{Current in } y\text{-direction: } d\hat{l} = dy'\hat{y}$$

   This problem actually consists of 4 different current segments that travel in two different directions. Two travel along x and two travel along y.

   (d) Pick limits of integration:

$$\oint_L \frac{Id\hat{l} \times (\vec{r} - \vec{r}')}{4\pi|\vec{r} - \vec{r}'|^3} = \underbrace{\int_{-L/2}^{L/2} \frac{Idx'\,\hat{x} \times (z\hat{z} - x'\hat{x} + \frac{L}{2}\hat{y})}{4\pi|z\hat{z} - x'\hat{x} + \frac{L}{2}\hat{y}|^3}}_{\text{Segment 1}} + \underbrace{\int_{-L/2}^{L/2} \frac{Idy'\,\hat{y} \times (z\hat{z} - \frac{L}{2}\hat{x} - y'\hat{y})}{4\pi|z\hat{z} - \frac{L}{2}\hat{x} - y'\hat{y}|^3}}_{\text{Segment 2}} +$$

$$\underbrace{\int_{L/2}^{-L/2} \frac{Idx'\,\hat{x} \times (z\hat{z} - x'\hat{x} - \frac{L}{2}\hat{y})}{4\pi|z\hat{z} - x'\hat{x} - \frac{L}{2}\hat{y}|^3}}_{\text{Segment 3}} + \underbrace{\int_{L/2}^{-L/2} \frac{Idy'\,\hat{y} \times (z\hat{z} + \frac{L}{2}\hat{x} - y'\hat{y})}{4\pi|z\hat{z} + \frac{L}{2}\hat{x} - y'\hat{y}|^3}}_{\text{Segment 4}}$$

   For this problem, our integral breaks into four pieces.

(e) Apply any appropriate symmetry arguments. This greatly helps on most work-out problems, particularly this one. If all 4 current segments are equal, then there should be no x or y components of field along the z axis. The two x-aligned segments will produce equal and opposite magnetic fields in the y-direction and the two y-aligned segments will produce equal and opposite magnetic fields in the x-direction. All four, however, will contribute equal amounts in the z-direction. Thus, we could write:

$$\vec{H}(0,0,z) = H_z(z)\hat{z} \qquad H_z = 4\hat{z}\cdot \underbrace{\int_{-L/2}^{L/2} \frac{I dx'\,\hat{x}\times(z\hat{z} - x'\hat{x} + \frac{L}{2}\hat{y})}{4\pi|z\hat{z} - x'\hat{x} + \frac{L}{2}\hat{y}|^3}}_{\text{Segment 1}}$$

(f) Simplify Integral:

$$\begin{aligned} H_z &= 4\hat{z}\cdot \underbrace{\int_{-L/2}^{L/2} \frac{I dx'\,\hat{x}\times(z\hat{z} - x'\hat{x} + \frac{L}{2}\hat{y})}{4\pi|z\hat{z} - x'\hat{x} + \frac{L}{2}\hat{y}|^3}}_{\text{Segment 1}} \\ &= \frac{IL}{2\pi}\int_{-L/2}^{L/2} \frac{dx'}{\left(z^2 + \frac{L^2}{4} + x'^2\right)^{\frac{3}{2}}} \\ &= \frac{IL}{2\pi\left(z^2 + \frac{L^2}{4}\right)} \left.\frac{x'}{\sqrt{z^2 + \frac{L^2}{4} + x'^2}}\right|_{x'=-\frac{L}{2}}^{x'=\frac{L}{2}} \\ &= \frac{IL}{2\pi\left(z^2 + \frac{L^2}{4}\right)} \frac{L}{\sqrt{z^2 + \frac{L^2}{2}}} \end{aligned}$$

After all the simplifications, the final answer is

$$\vec{H}(0,0,z) = \frac{I}{2\pi\left(\frac{z^2}{L^2} + \frac{1}{4}\right)\sqrt{z^2 + \frac{L^2}{2}}}\hat{z}$$

What a beautiful result! It took some math, but this problem is more difficult than any problem on the test.