ECE 3025 Homework 11: Faraday’s Law

Solutions

1. (a) Total flux, $\psi_M$, is

$$\psi_M = L^2 \vec{B} \cdot \hat{z} = 0.075 \cos(120 \pi t - 30^\circ)$$

$$I(t) = \frac{\text{emf}}{R} = -0.0565 \sin(120 \pi t - 30^\circ) \text{ Amps}$$

(b) This $\vec{B}$-field is not uniform across the square loop, so we have to integrate:

$$\psi_M = \int_0^L \int_0^L 0.4 \times 10^{-6} \cos(\pi ct - y) \, dx \, dy = 0.4 \times 10^{-6} \frac{L}{\pi} \left[ \sin(\pi ct - L) - \sin(\pi ct) \right]$$

$$= 6.4 \times 10^{-8} [\cos(\pi ct) - \sin(\pi ct)] = 9.0 \times 10^{-8} \cos \left( \pi ct - \frac{\pi}{4} \right)$$

$$I(t) = \frac{\text{emf}}{R} = -0.12 \sin \left( \pi ct - \frac{\pi}{4} \right) \text{ Amps}$$

2. (a) The current is due to the emf caused by the changing magnetic flux enclosed by the moving bar:

$$I(t) = \frac{\text{emf}}{R} = \frac{|\vec{B}| \text{ Area}}{2(2.2)(2 + 9t) + .3} = \frac{1.44}{9.1 + 39.6t} \text{ Amps}$$

where $R$ is the resistance of the two rails of length $2+9t$ and resistance $2.2 \ \Omega/m$ in series with the end resistor.

(b) Now there is a circuit on either side and two changing flux enclosed by the moving bar:

$$I(t) = \frac{\text{emf}}{R} = \frac{\frac{d}{dt}(0.8)(0.2)(2 + 9t) - \frac{d}{dt}(0.8)(0.2)(14 - 9t)}{R} = \frac{2.88}{R} \text{ Amps}$$

Now $R$ is two resistances in parallel. The right-hand side loop resistance consists of two rail lengths of $14-9t$ and an end resistance of $.3 \ \Omega$. The left-hand side loop resistance consists of two rail lengths of $2+9t$ and an end resistance of $.3 \ \Omega$. Thus,

$$R = R_L || R_R \quad R_R = 61.9 - 39.6t \quad R_L = 8.8 + 39.6t$$
3. Let us place one wire centered on the z-axis and one wire centered at \( x = b \) and \( y = 0 \). The first wire will carry a static current, \( I \), in the \( \hat{z} \) direction. The second wire will carry the same current in the \( -\hat{z} \) direction.

The total magnetic field around the wires along the \( x \)-axis is

\[
\vec{H}(x, 0, 0) = \left[ \frac{I}{2\pi x} + \frac{I}{2\pi (b-x)} \right] \hat{y}
\]

Now let us integrate the flux between the wires to estimate the per-unit-length inductance of the two lines:

\[
\Psi_M = \int_0^{1a} \int_a^{(b-a)} \vec{B}(x, 0, 0) \cdot dxdz \hat{y} = \frac{\mu I}{\pi} \ln \left( \frac{b-a}{a} \right)
\]

Thus, the per-unit-length inductance is given by

\[
L = \frac{\Psi_M}{I} = \frac{\mu}{\pi} \ln \left( \frac{b-a}{a} \right)
\]

If a uniform line charge, \( \rho_L \), is distributed along these same wires with positive polarity on the first wire and negative polarity on the second wire, the electric field is given by:

\[
\vec{E}(x, 0, 0) = \left[ \frac{\rho_L}{2\pi \epsilon x} + \frac{\rho_L}{2\pi \epsilon (b-x)} \right] \hat{x}
\]

Now let us integrate the voltage between these two wires:

\[
V = \int_a^{b-a} \vec{E}(x, 0, 0) \cdot dxd\hat{x} = \frac{\rho_L}{\pi \epsilon} \ln \left( \frac{b-a}{a} \right)
\]

Thus, the per-unit-length capacitance is given by

\[
C = \frac{\rho_L}{V} = \frac{\pi \epsilon}{\ln \left( \frac{b-a}{a} \right)}
\]

Now that we know the electrical properties of the transmission line, we can calculate the other physical parameters:

\[
Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{\pi} \sqrt{\frac{\mu}{\epsilon \ln \left( \frac{b-a}{a} \right)}} \quad \nu_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu \epsilon}}
\]
4. Inductive RFID Analysis.

(a) Let us use the result from the previous homework:

\[ \vec{H}(0, 0, z) = \frac{N_c I}{2\pi \left( \frac{z^2}{L^2} + \frac{1}{4} \right) \sqrt{z^2 + \frac{L^2}{2}}} \]

Plotting \( ||\vec{H}(0, 0, z)||^2/||\vec{H}(0, 0, z)||^2 \) produces

\[ \text{Power vs. Range in Square Loop} \]

Notice that when the card moves more than \( L \) away from the reader coil, the energy density in the static magnetic field is reduced by a factor of 100! This is why inductive RFID technology has such a limited range.

(b) In free space along the \( z \)-axis:

\[ \vec{B}(0, 0, z) = \frac{N_c I \mu_0}{2\pi \left( \frac{z^2}{L^2} + \frac{1}{4} \right) \sqrt{z^2 + \frac{L^2}{2}}} \]

For a card with \( N_c \) turns, the total magnetic flux is approximately:

\[ \Psi_{21} = N_c A \| \vec{B}(0, 0, z) \| = \frac{N_c N_c A I \mu_0}{2\pi \left( \frac{z^2}{L^2} + \frac{1}{4} \right) \sqrt{z^2 + \frac{L^2}{2}}} \]

where \( A \) is the card area (approximately 0.0015 m\(^2\) from the diagrams provided). Thus,
mutual inductance in this system is

\[ M = \frac{\Psi_{21}}{I} = \frac{N_r N_c A \mu_0}{2\pi \left( \frac{z^2}{r^2} + \frac{1}{4} \right) \sqrt{z^2 + \frac{L^2}{r^2}}}. \]

(c) For a card that is 5cm away from the reader, mutual inductance evaluates \( M = 831 \) nH/m. In class, we derived the Thevenin equivalent for two mutually-coupled circuits:

\[
\tilde{Z}_{Th} = j 2\pi f L_1 + \frac{4\pi^2 f^2 M^2}{\tilde{Z}_L + j 2\pi f L_2}, \quad \tilde{Z}_L = R + \frac{1}{j 2\pi f C}
\]

Without the card present, the Thevenin equivalent is \( \tilde{Z}_{Th} = j 3410 \Omega \). With the card present, the Thevenin equivalent is \( \tilde{Z}_{Th} = 200 + j 3410 \Omega \).