ECE 3025 Homework 11: Faraday's Law

Solutions

1. (a) Total flux, ψ_M, is

$$\Psi_M = L^2 \vec{B} \cdot \hat{z} = 0.075 \cos(120\pi t - 30^\circ)$$

$$I(t) = \frac{\overbrace{\frac{d\Psi_M}{dt}}^{\text{emf}}}{R} = -0.0565 \sin(120\pi t - 30^\circ) \text{ Amps}$$

   (b) This $\vec{B}$-field is not uniform across the square loop, so we have to integrate:

$$\Psi_M = \int_0^L \int_0^L 0.4 \times 10^{-6} \cos(\pi[ct - y])\, dx\, dy = 0.4 \times 10^{-6} \frac{L}{\pi} [\sin(\pi[ct - L]) - \sin(\pi ct)]$$

$$= 6.4 \times 10^{-8} [\cos(\pi ct) - \sin(\pi ct)] = 9.0 \times 10^{-8} \cos\left(\pi ct - \frac{\pi}{4}\right)$$

$$I(t) = \frac{\overbrace{\frac{d\Psi_M}{dt}}^{\text{emf}}}{R} = -0.12 \sin\left(\pi ct - \frac{\pi}{4}\right) \text{ Amps}$$

2. (a) The current is due to the emf caused by the changing magnetic flux enclosed by the moving bar:

$$I(t) = \frac{\overbrace{\frac{d\Psi_M}{dt}}^{\text{emf}}}{R} = \frac{\frac{d}{dt} \overbrace{(0.8)}^{|\vec{B}|} \overbrace{(0.2)(2 + 9t)}^{\text{Area}}}{2(2.2)(2 + 9t) + .3} = \frac{1.44}{9.1 + 39.6t} \text{Amps}$$

   where R is the resistance of the two rails of length 2+9t and resistance 2.2 Ω/m in series with the end resistor.

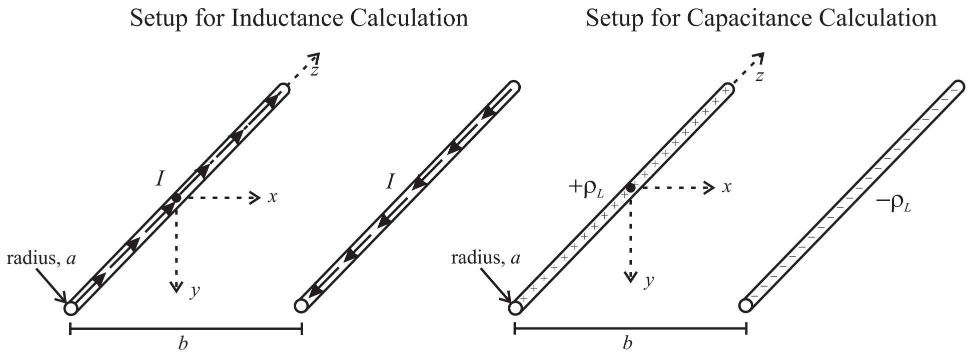
   (b) Now there is a circuit on either side and two changing flux enclosed by the moving bar:

$$I(t) = \frac{\frac{d\Psi_{ML}}{dt} - \frac{d\Psi_{MR}}{dt}}{R} = \frac{\frac{d}{dt}(0.8)(0.2)(2 + 9t) - \frac{d}{dt}(0.8)(0.2)(14 - 9t)}{R} = \frac{2.88}{R} \text{Amps}$$

   Now R is two resistances in parallel. The right-hand side loop resistance consists of two rail lengths of 14-9t and an end resistance of .3 Ω. The left-hand side loop resistance consists of two rail lengths of 2+9t and an end resistance of .3 Ω. Thus,

$$R = R_R || R_L \qquad R_R = 61.9 - 39.6t \qquad R_L = 8.8 + 39.6t$$

3. Let us place one wire centered on the z-axis and one wire centered at $x = b$ and $y = 0$. The first wire will carry a static current, I, in the $\hat{z}$ direction. The second wire will carry the same current in the $-\hat{z}$ direction.



The total magnetic field around the wires along the x-axis is

$$\vec{H}(x,0,0) = \left[\frac{I}{2\pi x} + \frac{I}{2\pi(b-x)}\right]\hat{y}$$

Now let us integrate the flux between the wires to estimate the per-unit-length inductance of the two lines:

$$\Psi_M = \int_0^{1\text{m}} \int_a^{(b-a)} \vec{B}(x,0,0)\cdot dx\,dz\,\hat{y} = \frac{\mu I}{\pi}\ln\left(\frac{b-a}{a}\right)$$

Thus, the per-unit-length inductance is given by

$$L = \frac{\Psi_M}{I} = \frac{\mu}{\pi}\ln\left(\frac{b-a}{a}\right)$$

If a uniform line charge, ρ_L, is distributed along these same wires with positive polarity on the first wire and negative polarity on the second wire, the electric field is given by:

$$\vec{E}(x,0,0) = \left[\frac{\rho_L}{2\pi\epsilon x} + \frac{\rho_L}{2\pi\epsilon(b-x)}\right]\hat{x}$$

Now let us integrate the voltage between these two wires:

$$V = \int_a^{b-a} \vec{E}(x,0,0)\cdot dx\,\hat{x} = \frac{\rho_L}{\pi\epsilon}\ln\left(\frac{b-a}{a}\right)$$

Thus, the per-unit-length capacitance is given by

$$C = \frac{\rho_L}{V} = \frac{\pi\epsilon}{\ln\left(\frac{b-a}{a}\right)}$$

Now that we know the electrical properties of the transmission line, we can calculate the other physical parameters:

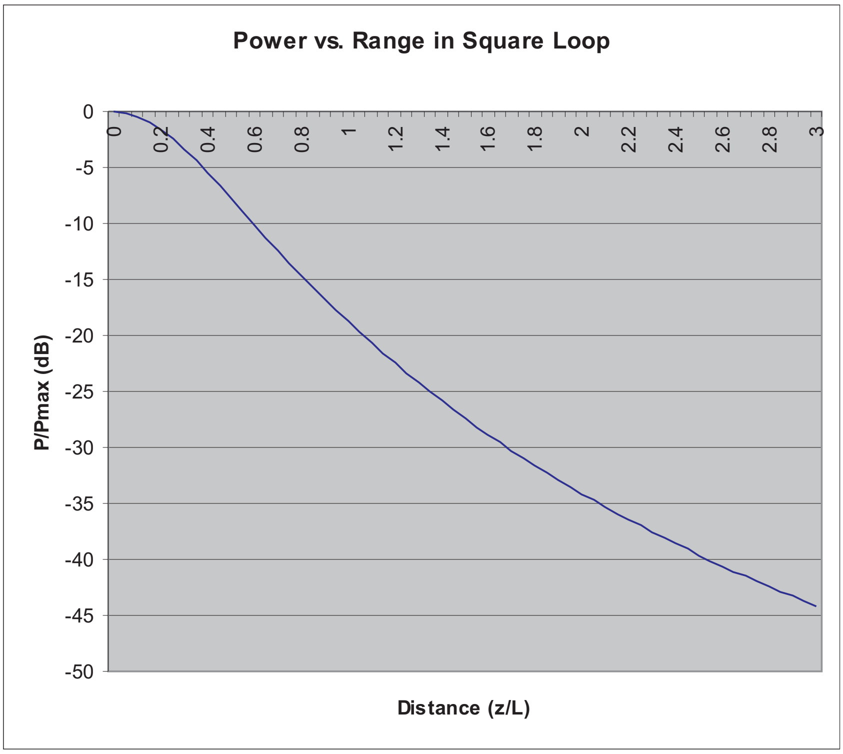
$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b-a}{a}\right) \qquad v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}}$$

4. Inductive RFID Analysis.

(a) Let us use the result form the previous homework:

$$\vec{H}(0,0,z) = \frac{N_r I}{2\pi\left(\frac{z^2}{L^2}+\frac{1}{4}\right)\sqrt{z^2+\frac{L^2}{2}}}\hat{z}$$

Plotting $\|\vec{H}(0,0,z)\|^2/\|\vec{H}(0,0,z)\|^2$ produces



Notice that when the card moves more than L away from the reader coil, the energy density in the static magnetic field is reduced by a factor of 100! This is why inductive RFID technology has such a limited range.

(b) In free space along the z-axis:

$$\vec{B}(0,0,z) = \frac{N_r I \mu_0}{2\pi\left(\frac{z^2}{L^2}+\frac{1}{4}\right)\sqrt{z^2+\frac{L^2}{2}}}\hat{z}$$

For a card with N_c turns, the total magnetic flux is approximately:

$$\Psi_{21} = N_c A\|\vec{B}(0,0,z)\| = \frac{N_r N_c A I \mu_0}{2\pi\left(\frac{z^2}{L^2}+\frac{1}{4}\right)\sqrt{z^2+\frac{L^2}{2}}}$$

where A is the card area (approximately 0.0015 m^2 from the diagrams provided). Thus, mutual inductance in this system is

$$M = \frac{\Psi_{21}}{I} = \frac{N_r N_c A \mu_0}{2\pi \left(\frac{z^2}{L^2} + \frac{1}{4}\right) \sqrt{z^2 + \frac{L^2}{2}}}$$

(c) For a card that is 5cm away from the reader, mutual inductance evaluates $M = 831$ nH/m. In class, we derived the Thevenin equivalent for two mutually-coupled circuits:

$$\tilde{Z}_{Th} = j2\pi f L_1 + \frac{4\pi^2 f^2 M^2}{\tilde{Z}_L + j2\pi f L_2} \qquad \tilde{Z}_L = R_{IC} + \frac{1}{j2\pi f C}$$

Without the card present, the Thevenin equivalent is $\tilde{Z}_{Th} = j3410\Omega$. With the card present, the Thevenin equivalent is $\tilde{Z}_{Th} = 200 + j3410\Omega$.