ECE 3025 Homework 2: Reflections on Transmission Lines

Solutions

1. When the pulse hits the junction of lines 1 and 2, it transmits with coefficient τ_{01} into line 2. When the pulse hits the junction of lines 2 and 3, it transmits with coefficient τ_{12} into line 3. Mathematically, these transmission coefficients are

$$\tau_{01} = \frac{2Z_1}{Z_0 + Z_1} \qquad \tau_{12} = \frac{2Z_2}{Z_1 + Z_2}$$

To figure out the maximum voltage, we maximize the product $\tau_{01}\tau_{12}$ with respect to Z_1. The value of impedance that maximizes the voltage amplitude is

$$Z_1 = \sqrt{Z_0 Z_2}$$

2. Key: 1-2-2 is shorthand for a signal with leading edge that travels down transmission line 1 (T1 transit time), then transmission line 2 (back and forth for +2T2 transit time), and finally down transmission line 1 again to the load (+T2 transit time).

   1.7 ns (1-2),
   2.7 ns (1-2-2),
   3.7 ns (1-2-2-2),
   4.1 ns (1-1-2),
   4.7 ns (1-2-2-2-2),
   5.1 ns (1-2-1-2 and 1-1-2-2),
   5.7 ns (1-2-2-2-2-2),
   6.1 ns (1-1-2-2-2 and 1-2-1-2-2),
   6.7 ns (1-2-2-2-2-2-2),
   7.1 ns (1-1-2-2-2-2 and 1-2-1-2-2-2),
   7.5 ns (1-2-1-1-2),
   7.7 ns (1-2-2-2-2-2-2-2)

3. (a) The initial excitation follows from the voltage divider equation:

$$V_{in} = V_S \frac{Z_0}{R_G + Z_0}$$

   (b) The answer to this question follows algebraically from our formulas:

$$\begin{aligned}
\gamma_L &= \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_G} \\
&= \frac{1 + \frac{R_L - Z_0}{R_L + Z_0}}{1 - \left(\frac{R_L - Z_0}{R_L + Z_0}\right)\left(\frac{R_G - Z_0}{R_G + Z_0}\right)} \\
&= \left[\frac{1 + \frac{R_L - Z_0}{R_L + Z_0}}{1 - \left(\frac{R_L - Z_0}{R_L + Z_0}\right)\left(\frac{R_G - Z_0}{R_G + Z_0}\right)}\right] \frac{R_G + Z_0}{R_G + Z_0}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2R_L}{R_L + Z_0 - (R_L - Z_0)\frac{R_G - Z_0}{R_G + Z_0}} \\
&= \frac{R_L(R_G + Z_0)}{Z_0(R_G + R_L)}
\end{aligned}
$$

If we plug in our expression in part (a) for V_{in} and simplify, we find that

$$V_L = V_S \frac{R_L}{R_G + R_L}$$

This is the simple voltage divider relationship that describes the steady-state DC circuit. As you can see, the complicated series of reflections on a transmission line always asymptotically tends towards the steady-state DC voltage – regardless of what values are used for R_G, R_L, and Z_0.

(c) Following the same procedure discussed in the problem statement, we can write this as the following summation:

$$V_{in}^{ss} = V_{in} + \underbrace{\Gamma_L(1+\Gamma_G)}_{\substack{\text{load reflection} \\ \text{source transmission}}} V_{in} + \underbrace{\Gamma_L\Gamma_G\Gamma_L(1+\Gamma_G)}_{\substack{\text{load reflection} \\ \text{source reflection} \\ \text{load reflection} \\ \text{source transmission}}} V_{in} + \cdots$$

where the first term is due to the initial excitation, the second term is a reflection from the load that is transmitted to the source, the third term is a reflection from the load, then the source, then the load, and a transmission to the source, and so on. Using our geometric series formula, we can write this series as

$$
\begin{aligned}
V_{in}^{ss} &= V_{in} + \Gamma_L(1+\Gamma_G)\sum_{n=0}^{\infty}\Gamma_L^n\Gamma_G^n V_{in} \\
&= V_{in} + \frac{\Gamma_L(1+\Gamma_G)}{1-\Gamma_L\Gamma_G}V_{in} \\
&= \frac{1+\Gamma_L}{1-\Gamma_L\Gamma_G}V_{in}
\end{aligned}
$$

Isn't that interesting! When we sum reflections from the source side, we get the same steady state transmission coefficient as if we summed reflections from the load side – even though the combinations of physical reflections are different. Overall this should be very comforting: we have been saying all along that a lossless transmission line acts like a short circuit to the load resistance *after reaching a DC steady state.* Now we've proven it.

(d) Combine these two results

$$V_{in}^{ss} = \frac{1+\Gamma_L}{1-\Gamma_L\Gamma_G}V_{in} = \frac{R_L(R_G+Z_0)}{Z_0(R_G+R_L)}V_{in}$$

$$V_{in} = V_S\frac{Z_0}{R_G+Z_0}$$

to arrive at

$$V_{in}^{ss} = \frac{R_L(R_G+Z_0)}{Z_0(R_G+R_L)}\frac{Z_0}{R_G+Z_0}V_S = \frac{R_L}{R_G+R_L}V_S$$

This describes the behavior of a simple voltage divider circuit between R_L and R_G, taken across the load resistor – just as we covered in class.

(e) At first, the DC-excited transmission line appears to be a resistor of value Z_0. The initial voltage and current charges the line through a series of reflections and transmissions until the line reaches a steady state. Then the transmission line appears to be a resistor of value R_L.