ECE 3025 Homework 7: Fields and Charges

Solutions

1. Setting up E-Field Integrals:

   (a)

$$\vec{E}(\vec{r}) = \int\limits_S \frac{\overbrace{\rho_S(\vec{r}')}^{\rho_S}(\vec{r}-\vec{r}')\overbrace{dS}^{\rho\, d\rho\, d\phi}}{4\pi\epsilon||\vec{r}-\vec{r}'||^3}$$

$$\vec{E}(x\hat{x}+y\hat{y}+z\hat{z}) = \int_0^R\int_0^{2\pi} \frac{\rho_S(\overbrace{x\hat{x}+y\hat{y}+z\hat{z}}^{\vec{r}}\overbrace{-\rho\cos\phi\hat{x}-\rho\sin\phi\hat{y}-0\hat{z}}^{-\vec{r}'})\rho\, d\rho\, d\phi}{4\pi\epsilon||\underbrace{x\hat{x}+y\hat{y}+z\hat{z}}_{\vec{r}}\underbrace{-\rho\cos\phi\hat{x}-\rho\sin\phi\hat{y}-0\hat{z}}_{-\vec{r}'}||^3}$$

$$\vec{E}(x,y,z) = \frac{\rho_S}{4\pi\epsilon}\int_0^R\int_0^{2\pi} \frac{[(x-\rho\cos\phi)\hat{x}+(y-\rho\sin\phi)\hat{y}+z\hat{z}]\ \rho\, d\rho\, d\phi}{[(x-\rho\cos\phi)^2+(y-\rho\sin\phi)^2+z^2]^{\frac{3}{2}}}$$

   (b)

$$\vec{E}(\vec{r}) = \int\limits_S \frac{\overbrace{\rho_S(\vec{r}')}^{\rho_0\sqrt{1-(z')^2}}\ (\vec{r}-\vec{r}')\ \overbrace{dS}^{r^2\sin\theta\, d\theta\, d\phi}}{4\pi\epsilon||\vec{r}-\vec{r}'||^3}$$

$$\vec{E}(x,y,z) = \frac{\rho_0}{4\pi\epsilon}\int_0^{\pi}\int_0^{2\pi} \frac{\sqrt{1-(z')^2}(\overbrace{x\hat{x}+y\hat{y}+z\hat{z}}^{\vec{r}}\overbrace{-r\sin\theta\cos\phi\hat{x}-r\sin\theta\sin\phi\hat{y}-r\cos\theta\hat{z}}^{-\vec{r}'})r^2\sin\theta\, d\theta\, d\phi}{||\underbrace{x\hat{x}+y\hat{y}+z\hat{z}}_{\vec{r}}\underbrace{-\rho\cos\phi\hat{x}-\rho\sin\phi\hat{y}-0\hat{z}}_{-\vec{r}'}||^3}$$

$$= \frac{\rho_0}{4\pi\epsilon}\int_0^{\pi}\int_0^{2\pi} \frac{\sqrt{1-\cos^2\theta}\,[(x-\sin\theta\cos\phi)\hat{x}+(y-\sin\theta\sin\phi)\hat{y}+(z-\cos\theta)\hat{z}]}{[(x-\sin\theta\cos\phi)^2+(y-\sin\theta\sin\phi)^2+(z-\cos\theta)^2]^{\frac{3}{2}}}\sin\theta\, d\theta\, d\phi$$

$$= \frac{\rho_0}{4\pi\epsilon}\int_0^{\pi}\int_0^{2\pi} \frac{[(x-\sin\theta\cos\phi)\hat{x}+(y-\sin\theta\sin\phi)\hat{y}+(z-\cos\theta)\hat{z}]\sin^2\theta\, d\theta\, d\phi}{[(x-\sin\theta\cos\phi)^2+(y-\sin\theta\sin\phi)^2+(z-\cos\theta)^2]^{\frac{3}{2}}}$$

   (c) Since this line charge spirals in space and is not aligned along any particular axis, it is easiest to integrate the curve over the parameter t. What we must realize first, however, is that $dL = \sqrt{R^2+1}dt$. In other words, for every 1 unit of t advanced, we advance $\sqrt{R^2+1}$ units of physical distance along the helix.

$$\vec{E}(\vec{r}) = \int_{-\infty}^{+\infty} \frac{\overbrace{\rho_L(\vec{r}')}^{\rho_L}(\vec{r}-\vec{r}')\overbrace{dL}^{\sqrt{R^2+1}dt}}{4\pi\epsilon||\vec{r}-\vec{r}'||^3}$$

$$\begin{aligned}
\vec{E}(x,y,z) &= \frac{\rho_L\sqrt{R^2+1}}{4\pi\epsilon}\int_{-\infty}^{+\infty}\frac{(\overbrace{x\hat{x}+y\hat{y}+z\hat{z}}^{\vec{r}}\overbrace{-x'(t)\hat{x}-y'(t)\hat{y}-z'(t)\hat{z}}^{-\vec{r}'})dt}{||\underbrace{x\hat{x}+y\hat{y}+z\hat{z}}_{\vec{r}}\underbrace{-x'(t)\hat{x}-y'(t)\hat{y}-z'(t)\hat{z}}_{-\vec{r}'}||^3} \\
&= \frac{\rho_L\sqrt{R^2+1}}{4\pi\epsilon}\int_{-\infty}^{+\infty}\frac{(x\hat{x}+y\hat{y}+z\hat{z}-R\cos t\hat{x}-R\sin t\hat{y}-t\hat{z})dt}{||x\hat{x}+y\hat{y}+z\hat{z}-R\cos t\hat{x}-R\sin t\hat{y}-t\hat{z}||^3} \\
&= \frac{\rho_L\sqrt{R^2+1}}{4\pi\epsilon}\int_{-\infty}^{+\infty}\frac{[(x-R\cos t)\hat{x}+(y-R\sin t)\hat{y}+(z-t)\hat{z}]\ dt}{[(x-R\cos t)^2+(y-R\sin t)^2+(z-t)^2]^{\frac{3}{2}}}
\end{aligned}$$

As a side note, if you let $R \to 0$, this last integral should become the solution to the infinite line charge problem discussed in class.

2. For a point of observation $\vec{P} = (8, 12, 2)$ and a point of charge $\vec{A} = (4, 3, 5)$:

$$\vec{E} = \frac{Q(\vec{P}-\vec{A})}{4\pi\epsilon_0||\vec{P}-\vec{A}||^3} = 16.5(4\hat{x}+9\hat{y}-3\hat{z}) \text{ Volts/m}$$

The point of observation (8,12,2) has the following cylindrical coordinate parameters:

$$\rho = 14.4 \qquad \phi = 0.9828 \text{ R} \qquad z = 2.0$$

When plugged into the conversion formulas, the field at this point is

$$\vec{E} = 160.2\hat{a}_\rho + 27.5\hat{a}_\phi - 49.5\hat{z} \text{ Volts/m}$$

These conversion problems can get kind of complicated, but a quick sanity check is to take the norm of $\vec{E}$ in both Cartesian and Cylindrical or Spherical coordinates; the magnitudes should be identical in any coordinate system.

3. The total charge, Q, is equal to

$$\begin{aligned}
Q &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_v(x,y,z)\,dx\,dy\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-|x|-|y|-|z|)\,dx\,dy\,dz \\
&= 2\int_0^{\infty}\exp(-x)\,dx\ 2\int_0^{\infty}\exp(-y)\,dy\ 2\int_0^{\infty}\exp(-z)\,dz \\
&= 8 \text{ C}
\end{aligned}$$

4. We know how to solve for the line charge if it rests on the z-axis:

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0(x^2 + y^2)}[x\hat{x} + y\hat{y}]$$

This is a basic adaptation from the formula in the book. If the line of charge is shifted to $y = y_0$, then the modified expression becomes:

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0[x^2 + (y - y_0)^2]}[x\hat{x} + (y - y_0)\hat{y}]$$

Thus, total field for the *two* lines in this problem is given by

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0[x^2 + (y - y_0)^2]}[x\hat{x} + (y - y_0)\hat{y}] - \frac{\rho_L}{2\pi\epsilon_0[x^2 + (y + y_0)^2]}[x\hat{x} + (y + y_0)\hat{y}]$$

For the first observation point in part (a), $(x, 0, z)$, this expression becomes

$$\vec{E} = -\frac{\rho_L y_0}{\pi\epsilon_0[x^2 + y_0^2]}\hat{y} = \frac{-8632}{x^2 + 0.36}\hat{y} \text{ Volts/m}$$

For the second observation point in part (b), $(2, 3, 4)$, this expression becomes

$$\vec{E} = \frac{7193}{[2^2 + 2.4^2]}[2\hat{x} + 2.4\hat{y}] - \frac{7193}{[2^2 + 3.6^2]}[2\hat{x} + 3.6\hat{y}] = 626\hat{x} + 242\hat{y} \text{ Volts/m}$$

Note that there is no dependence on z in this field.

5. From the previous problem, we know that the E-field from the first line of charge is going to be

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0[x^2 + (y - y_0)^2]}[x\hat{x} + (y - y_0)\hat{y}] = \frac{1349}{[x^2 + (y - .4)^2]}[x\hat{x} + (y - .4)\hat{y}]$$

The point of observation is the other line, which can be represented as $(0, -.4, z)$, and leads to a field of:

$$\vec{E} = \frac{1349}{[0^2 + (-.4 - .4)^2]}[0\hat{x} + (-.4 - .4)\hat{y}] = -1686\hat{y} \text{ Volts/m}$$

To get force per unit meter that line 1 exerts on line 2, we multiply this field by the charge density of the other line:

$$\vec{F} = \rho_L\vec{E} = -1.64 \times 10^{-4}\hat{y} \text{ Newtons/m}$$

The force that line 2 exerts on line 1 would be equal and opposite: $+1.64 \times 10^{-4}\hat{y}$ Newtons/m.