ECE 3025 Homework 8: Voltage Problems

Solutions

1. **Integral Set-up:**

(a)

$$
\begin{aligned}
V(\vec{r}) &= \int\limits_{\text{vol}} \frac{\overbrace{\rho_V(\vec{r}')}^{\rho_V}\ \overbrace{dv}^{dx'dy'dz'}}{4\pi\epsilon|\vec{r}-\vec{r}'|} \\
V(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z) &= \int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\rho_V\, dx'dy'dz'}{4\pi\epsilon|\underbrace{x\hat{a}_x + y\hat{a}_y + z\hat{a}_z}_{\vec{r}}\underbrace{-x'\hat{a}_x - y'\hat{a}_y - z'\hat{a}_z}_{-\vec{r}'}|} \\
V(x,y,z) &= \frac{\rho_V}{4\pi\epsilon}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V(\vec{r}) &= \overbrace{\int\limits_{S_1}\frac{\rho_S(\vec{r}_1')\,ds_1}{4\pi\epsilon|\vec{r}-\vec{r}_1'|}}^{\text{Top Plate}} + \overbrace{\int\limits_{S_2}\frac{\rho_S(\vec{r}_2')\,ds_2}{4\pi\epsilon|\vec{r}-\vec{r}_2'|}}^{\text{Bottom Plate}} \\
V(x,y,z) &= \int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\frac{-\rho_S\,dx_1'dy_1'}{4\pi\epsilon|\vec{r}-\vec{r}_1'|} + \int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\frac{\rho_S\,dx_2'dy_2'}{4\pi\epsilon|\vec{r}-\vec{r}_2'|} \\
V(x,y,z) &= -\frac{\rho_S}{4\pi\epsilon}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx_1'dy_1'}{\sqrt{(x-x_1')^2+(y-y_1')^2+(z-d)^2}} + \frac{\rho_S}{4\pi\epsilon}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\int\limits_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dx_2'dy_2'}{\sqrt{(x-x_2')^2+(y-y_2')^2+z^2}}
\end{aligned}
$$

(c)

$$\vec{E}(\vec{r}) = \int_{S} \frac{\overbrace{\rho_S(\vec{r}')}^{\rho_0}(\vec{r}-\vec{r}')\,dS}{4\pi\epsilon||\vec{r}-\vec{r}'||^3}$$

$$\vec{E}(x,y,0) = \frac{\rho_0}{4\pi\epsilon}\int_{\substack{\text{Top}\\\text{Cone}}} \frac{(\overbrace{x\hat{x}+y\hat{y}+0\hat{z}}^{\vec{r}}\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{x\hat{x}+y\hat{y}+0\hat{z}}_{\vec{r}}\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3} - \frac{\rho_0}{4\pi\epsilon}\int_{\substack{\text{Bot.}\\\text{Cone}}} \frac{(\overbrace{x\hat{x}+y\hat{y}+0\hat{z}}^{\vec{r}}\overbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}^{-\vec{r}'})\,dS}{||\underbrace{x\hat{x}+y\hat{y}+0\hat{z}}_{\vec{r}}\underbrace{-x'\hat{x}-y'\hat{y}-z'\hat{z}}_{-\vec{r}'}||^3}$$

$$= \frac{\rho_0}{4\pi\epsilon}\int_a^b\int_0^{2\pi} \frac{[(x-r\sin(90^\circ-\alpha))\cos\phi)\hat{x} + (y-r\sin(90^\circ-\alpha))\sin\phi)\hat{y} + r\cos(90^\circ-\alpha))\hat{z}]}{[(x-r\sin(90^\circ-\alpha))\cos\phi)^2 + (y-r\sin(90^\circ-\alpha))\sin\phi)^2 + r^2\cos^2(90^\circ-\alpha))]^{\frac{3}{2}}} r\sin(90^\circ-\alpha))\,d\phi\,dr$$

$$- \frac{\rho_0}{4\pi\epsilon}\int_a^b\int_0^{2\pi} \frac{[(x-r\sin(90^\circ+\alpha)\cos\phi)\hat{x} + (y-r\sin(90^\circ+\alpha)\sin\phi)\hat{y} + r\cos(90^\circ+\alpha)\hat{z}]}{[(x-r\sin(90^\circ+\alpha)\cos\phi)^2 + (y-r\sin(90^\circ+\alpha)\sin\phi)^2 + r^2\cos^2(90^\circ+\alpha)]^{\frac{3}{2}}} r\sin(90^\circ+\alpha)\,d\phi\,dr$$

$$= \frac{\rho_0\cos\alpha}{4\pi\epsilon}\int_a^b\int_0^{2\pi} \frac{[(x-r\cos\alpha\cos\phi)\hat{x} + (y-r\cos\alpha\sin\phi)\hat{y} + r\sin\alpha\hat{z}]}{\left[(x-r\cos\alpha\cos\phi)^2 + (y-r\cos\alpha\sin\phi)^2 + r^2\sin^2\alpha\right]^{\frac{3}{2}}} r\,d\phi\,dr$$

$$- \frac{\rho_0\cos\alpha}{4\pi\epsilon}\int_a^b\int_0^{2\pi} \frac{[(x-r\cos\alpha\cos\phi)\hat{x} + (y-r\cos\alpha\sin\phi)\hat{y} - r\sin\alpha\hat{z}]}{\left[(x-r\cos\alpha\cos\phi)^2 + (y-r\cos\alpha\sin\phi)^2 + r^2\sin^2\alpha\right]^{\frac{3}{2}}} r\,d\phi\,dr$$

$$= \frac{\rho_0\sin(2\alpha)\hat{z}}{4\pi\epsilon}\int_a^b dr\int_0^{2\pi} d\phi \frac{r^2}{\left[(x-r\cos\alpha\cos\phi)^2 + (y-r\cos\alpha\sin\phi)^2 + r^2\sin^2\alpha\right]^{\frac{3}{2}}}$$

2. **Sphere Charge:** Start by noting that 1) this problem is rotationally symmetric about the origin (no dependence on the point of observation in terms of θ and ϕ) and 2) the flux density will only point in the $\hat{r}$ direction. Thus $\vec{D}(r,\phi,\theta) = D_r(r)\hat{r}$. Applying Gauss's Law, we derive two expressions for flux density vector through a spherical shell of radius r – one for points of observation outside the sphere (where enclosed charge is equal to sphere volume $\times$ charge density) and one for points of observation inside the sphere (where enclosed charge varies as function of r). The result is:

$$\vec{D}(r) = \begin{cases} \frac{4\pi r \rho_v}{3}\hat{r} & r < R \\ \\ \frac{4\pi R^3 \rho_v}{3r^2}\hat{r} & r \geq R \end{cases}$$

Now let's verify this with the divergence theorem. In spherical coordinates, the divergence of a $\vec{D}(r)$ that only has a component in the $\hat{r}$ direction is

$$\nabla \cdot \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) = \begin{cases} \rho_v & r < R \\ \\ 0 & r \geq R \end{cases}$$

which is consistent with the original problem statement: a uniform volume charge inside a sphere of radius R.