Name: ______________________________

GTID: ______________________________

ECE 3025: Electromagnetics

TEST 2 (Fall 2009)

- Please read all instructions before continuing with the test.
- This is a **closed** notes, **closed** book, **closed** calculator, **closed** friend, **open** mind test. You should only have writing instruments on your desk when you take this test. If I find anything on your desk (excluding the test itself, writing instruments, and life-or-death medication) I will turn you in for an honor code violation. I am serious.
- Show all work. (It helps me give partial credit.) Work all problems in the spaces below the problem statement. If you need more room, use the back of the page. DO NOT use or attach extra sheets of paper for work.
- Work intelligently – read through the exam and do the easiest problems first. Save the hard ones for last.
- All necessary mathematical formulas are included either in the problem statements or the last few pages of this test.
- You have 80 minutes to complete this examination. When I announce a "last call" for examination papers, I will leave the room in 5 minutes. The fact that I do not have your examination in my possession will not stop me.
- I will not grade your examination if you fail to 1) put your name and GTID number in the upper left-hand blanks on this page or 2) sign the blank below acknowledging the terms of this test and the honor code policy.
- Have a nice day!

Pledge Signature: ____

I acknowledge the above terms for taking this examination. I have neither given nor received unauthorized help on this test. I have followed the Georgia Tech honor code in preparing and submitting the test.

(1) **Lossy Line:** The power company is planning to run L meters of single-phase power distribution into a local neighborhood, fed with V_0 volts. If the attenuation constant for this section of line is α, write an expression for the magnitude of the total voltage at the end of this (matched) line. (**14 points**)

(2) **Coordinate Systems:** Answer the following questions about non-Cartesian 3D coordinate systems (**15 points**).

(a) Write expressions for the cylindrical coordinate unit vectors $\hat{\rho}$, $\hat{\phi}$, $\hat{z}$ in terms of Cartesian unit vectors for an observation point at $\rho = 6$, $\phi = 90°$, and $z = 7$.

(b) Write expressions for the spherical coordinate unit vectors $\hat{r}$, $\hat{\phi}$, $\hat{\theta}$ in terms of Cartesian unit vectors for an observation point at $r = 2$, $\phi = 180°$, and $\theta = 90°$.

(c) Write expressions for the spherical coordinate unit vectors $\hat{r}$, $\hat{\phi}$, $\hat{\theta}$ in terms of Cartesian unit vectors for an observation point at $r = 12$, $\phi = 0°$, and $\theta = 0°$.

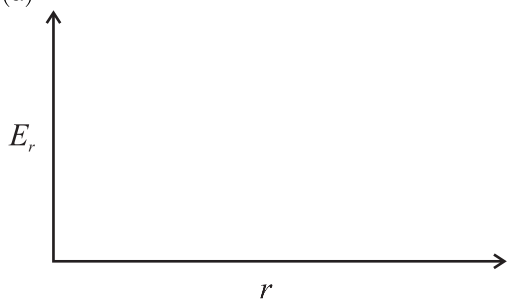
(3) **Electrostatic Charge Distributions:** All of the field distributions in this problem are free-space and may be written in the following form:

$$\vec{E}(r,\phi,\theta) = E_r(r)\hat{r}$$

Make a rough sketch in the graph provided of $E_r(r)$ for the following charge distributions. (**21 points**)

(a)

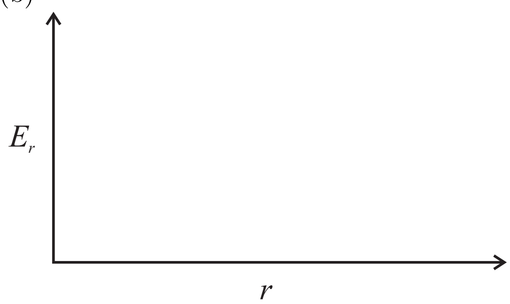
At the origin is a +1 C point charge. Surrounding this charge is a spherical shell of uniform charge density centered at the origin at a radius of R and a total charge of +1 C.



(b)

There is a charge distribution in space of the form:

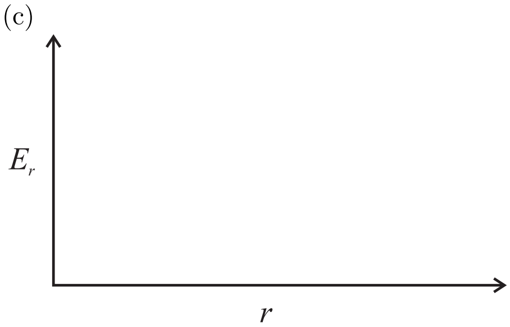
$$\rho_v(r,\phi,\theta) = \rho_o$$



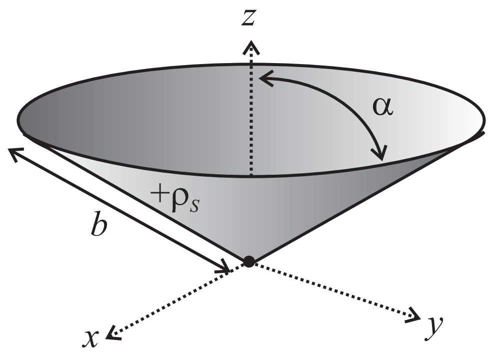
(c)

There is a voltage distribution in space of the form:

$$V(r,\phi,\theta) = V_o \exp(-r)$$



(4) **Electrostatic Integrals:** Below is a cone of uniform surface charge density, ρ_s, with tip at the origin extending out to a distance b, at an angle α to the z-axis. Answer all questions based on this scenario. (**50 points**).



(a) Calculate the total amount of charge on the cone, expressed in variables. Fully evaluate this answer. (**10 points**)

(b) Write an expression for the electric field observed along the z-axis, $\vec{E}(0,0,z)$, due to the presence of the charge cone. You must simplify as much as possible, but you do not need to evaluate the integrals. (**15 points**)

(c) Write an expression for the voltage on the xy-plane, $V(x, y, 0)$, due to the presence of the charge cone. You must simplify as much as possible, but you do not need to evaluate the integrals. (**15 points**)

(d) Your results in (b) and (c) are setup for efficient evaluation by a computer. One way to check numerical electromagnetic computations is to test limiting cases of the computation against canonical problems with known analytic solutions. In class, we derived the result for an infinite plane. Under what geometrical conditions of b and α would you be able to check your answer in part (b) against the infinite plane problem? (**5 points**)

(e) Along a similar vein, how might you use the result in (b) to test the results against the electric field of the canonical line charge problem we derived in class (uniform charge on the z-axis from $-a \leq z \leq a$)? (**5 points**)

Formula Sheet

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta D}{Z_0 + jZ_L \tan \beta D}$$

$$\alpha = \alpha_c + \alpha_d \, \text{(Nepers/m)} \qquad \text{(dB/m):} 8.7\alpha$$

$$\begin{aligned} v(z,t) &= V^+ f\left(t - \tfrac{z}{v_p}\right) + V^- g\left(t + \tfrac{z}{v_p}\right) & \tilde{v}(z) &= V^+ \exp(-\gamma z) + V^- \exp(+\gamma z) \\ i(z,t) &= \tfrac{V^+}{Z_0} f\left(t - \tfrac{z}{v_p}\right) - \tfrac{V^-}{Z_0} g\left(t + \tfrac{z}{v_p}\right) & \tilde{i}(z) &= \tfrac{V^+}{Z_0} \exp(-\gamma z) - \tfrac{V^-}{Z_0} \exp(+\gamma z) \end{aligned}$$

$$\vec{F} = Q\vec{E} \qquad \text{Point Charge at the Origin: } \vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

Charge Distributions: $$\vec{E}(\vec{r}) = \int\limits_L \frac{\rho_L(\vec{r}')(\vec{r} - \vec{r}')dL}{4\pi\epsilon||\vec{r} - \vec{r}'||^3} = \iint\limits_S \frac{\rho_S(\vec{r}')(\vec{r} - \vec{r}')dS}{4\pi\epsilon||\vec{r} - \vec{r}'||^3} = \iiint\limits_V \frac{\rho_V(\vec{r}')(\vec{r} - \vec{r}')dV}{4\pi\epsilon||\vec{r} - \vec{r}'||^3}$$

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon r} = \int\limits_L \frac{\rho_L(\vec{r}')dL}{4\pi\epsilon||\vec{r} - \vec{r}'||} = \iint\limits_S \frac{\rho_S(\vec{r}')dS}{4\pi\epsilon||\vec{r} - \vec{r}'||} = \iiint\limits_V \frac{\rho_V(\vec{r}')dV}{4\pi\epsilon||\vec{r} - \vec{r}'||}$$

Variable of integration: $\vec{r}'$ Point of observation: $\vec{r}$

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad \nabla \times \vec{E} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\vec{D} = \epsilon\vec{E} \qquad \vec{E} = -\nabla V \qquad \nabla \cdot \vec{D} = \rho_v \qquad V_B - V_A = -\int_A^B \vec{E} \cdot d\hat{l}$$

Cross Product: $$\vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \|\vec{a}\|\|\vec{b}\| \sin \theta_{ab} \hat{n}$$

Dot Product: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \|\vec{a}\|\|\vec{b}\| \cos \theta_{ab}$

Sphere Volume: $\dfrac{4\pi}{3} r^3$ Sphere Surface Area: $4\pi r^2$

Infinite Charge Sheet on xy-plane: $\vec{E} = \dfrac{\rho_o}{2\epsilon}\hat{z}$ (flip sign for $z < 0$) $= \dfrac{\rho_o}{2\epsilon}\text{sgn}(z)\hat{z}$

DIVERGENCE

CARTESIAN $\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

CYLINDRICAL $\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$

SPHERICAL $\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$

GRADIENT

CARTESIAN $\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$

CYLINDRICAL $\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$

SPHERICAL $\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$

CURL

CARTESIAN $\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$

CYLINDRICAL
$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right]\mathbf{a}_z$$

SPHERICAL
$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left[\frac{\partial(H_\phi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial \phi}\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\mathbf{a}_\phi$$

LAPLACIAN

CARTESIAN $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

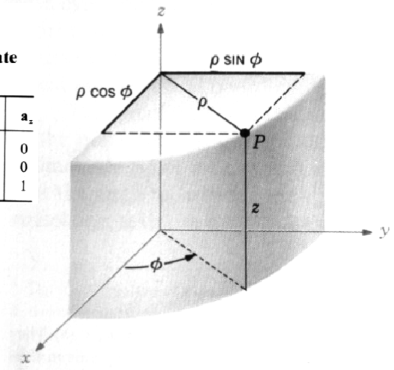
CYLINDRICAL $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$

SPHERICAL $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$

Cylindrical Coordinates

Dot products of unit vectors in cylindrical and cartesian coordinate systems

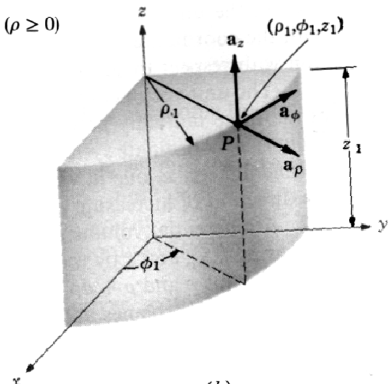
	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	1



$\rho = \sqrt{x^2 + y^2}$ $(\rho \geq 0)$

$\phi = \tan^{-1}\frac{y}{x}$

$z = z$

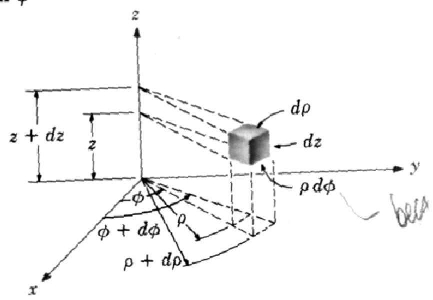


(b)

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

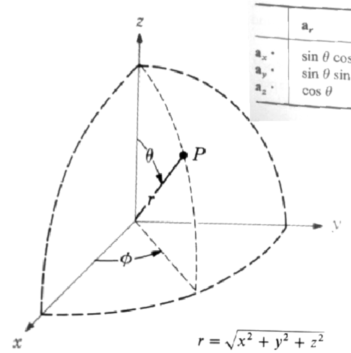


(c)

Spherical Coordinates

Dot products of unit vectors in spherical and cartesian coordinate systems

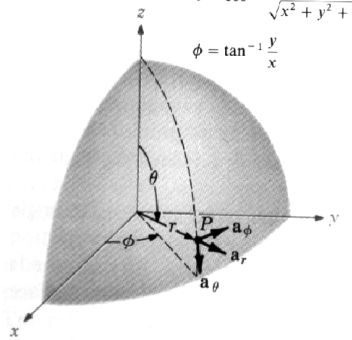
	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0



$r = \sqrt{x^2 + y^2 + z^2}$ $(r \geq 0)$

$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}$ $(0° \leq \theta \leq 180°)$

$\phi = \tan^{-1}\frac{y}{x}$



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$