ECE 3025: Electromagnetics

Solutions to TEST 3 (Fall 2005)

(1) **Laplace's Equation:**

(a) We know from symmetry that the voltage will not depend on z or ρ. Thus, applying Laplace's equations:

$$\nabla^2 V(\rho, \phi, z) = \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

First, multiply through by ρ^2 to get the simplified partial-differential equation:

$$\frac{\partial^2 V}{\partial \phi^2} = 0$$

Integrate this twice to produce the general solution:

$$V(\phi) = C_1 + C_2\phi$$

Enforce boundary conditions to solve for the constants C_1 and C_2:

$$V(0) = C_1 = 0V \qquad V(\phi) = C_2\pi = V_0$$

Thus the final solution may be written as

$$V(\rho, \phi, z) = \frac{V_0|\phi|}{\pi}$$

Note that, due to the symmetry of the problem, $V(\phi) = V(-\phi)$; thus, the solution we sketched out in $0 \le \phi \le \pi$ is valid on the other side of the xz-plane as well.

(b) Recognize that $\vec{E} = -\nabla V(\phi)$. The only difficulty in the problem is to make sure we use the gradient formula for the *cylindrical* coordinate system. For $0 \le \phi \le \pi$:

$$\vec{E}(\rho) = -\frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi}\hat{\phi} = -\frac{V_0}{\rho\pi}\hat{\phi}$$

Again, since the solution reflects about the xz-axis, we may write:

$$\vec{E}(\rho, \phi, z) = \begin{cases} -\frac{V_0}{\rho\pi}\hat{\phi} & 0 < \phi < \pi \\ \frac{V_0}{\rho\pi}\hat{\phi} & -\pi < \phi < 0 \end{cases} \quad \textbf{or} \quad \vec{E}(\rho, \phi, z) = \frac{2V_0}{\rho\pi}\left[\frac{1}{2} - \mathrm{u}(\phi)\right]\hat{\phi}$$

(c) To find charge from an electric field distribution, we apply Gauss's law in differential form: $\nabla \cdot \vec{D} = \rho_v$. Note that the divergence formula in cylindrical coordinates predicts that

$$\nabla \cdot (\epsilon\vec{E}) = \epsilon\frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} = 0$$

for $0 < \phi < \pi$. Does this make sense? Yes, because there is no charge in the space (according to Laplace's equations). The only charge in space exists as a surface charge on the two plates. The full solution is easiest to obtain by using the expression provided you in the previous problem:

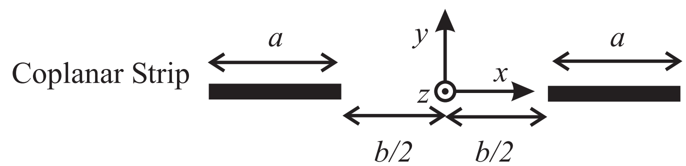
$$\rho_v = \epsilon\frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} = \frac{\epsilon}{\rho}\frac{\partial}{\partial \phi}\left\{\frac{2V_0}{\rho\pi}\left[\frac{1}{2} - \mathrm{u}(\phi)\right]\right\} = -\frac{2\epsilon V_0}{\pi\rho^2}\delta(\phi)$$

If you got to this point, you received full credit. Note, however that $\rho_v(\vec{r}) = \rho_s(\rho, z)\frac{\delta(\phi)}{\rho}$, which is how to translate a volume charge into a surface charge. Thus, the surface charge on each sheet is

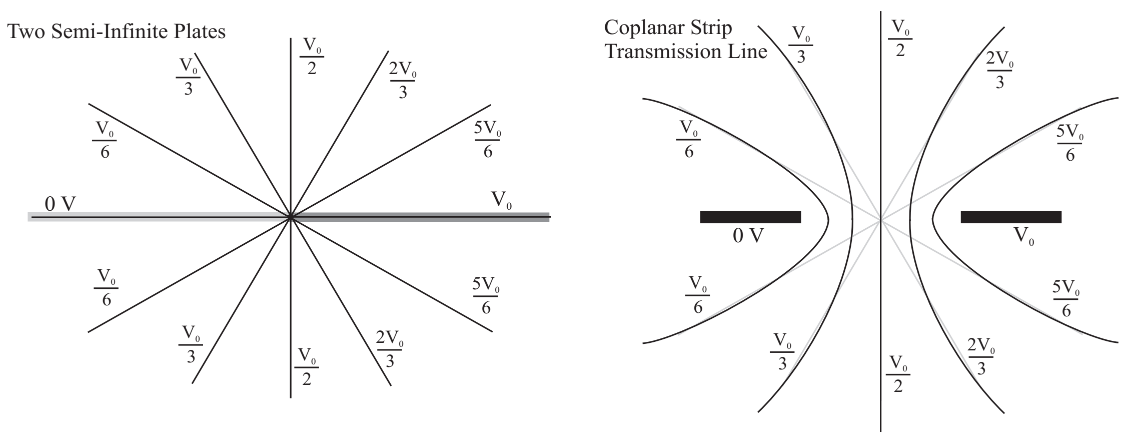
$$\rho_s = \pm\frac{2\epsilon V_0}{\pi\rho}$$

The positive density is on the 100V plate and the negative density is on the 0V plate. Note that the test formula $\rho_v(\vec{r}) = \rho_s(\rho, z)\delta(\phi)$ had a typo in it (no ρ in the denominator). Of course, anyone who omitted the ρ was still given full credit (not that it mattered at that point in the problem).

(d) **Bonus Challenge:** No one got this. Here is the procedure, for those interested. First, note that we should adapt the geometry of our simple charge plates to the geometry of the transmission line:



Compare a sketch of equipotential lines below for the two semi-infinite plates that you solved in this test problem to the coplanar strip transmission line with finite plates:



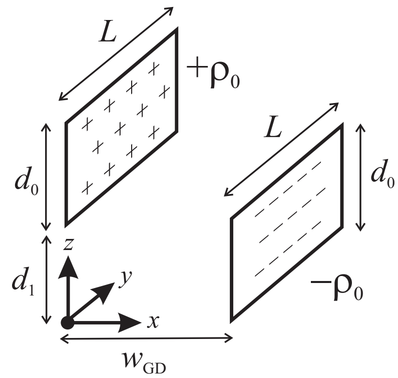
Your expressions are a pretty good approximation to the fields and charges in this very practical problem. The approximation only breaks down close to the origin. However, many engineers use the approximations of this problem to derive per-unit-length capacitance and inductance for coplanar strip transmission lines – particularly when b is small relative to a. If this is true, then per-unit-length capacitance is given by

$$C = \frac{\rho_L}{V_0} = \frac{1}{V_0}\int_{\frac{b}{2}}^{\frac{b}{2}+a} \rho_s\, d\rho = \int_{\frac{b}{2}}^{\frac{b}{2}+a} \frac{2\epsilon V_0\, d\rho}{\pi\rho} = \frac{2\epsilon}{\pi}\ln\left(1 + \frac{2a}{b}\right)$$

Utter sweetness!

(2) **MOSFET Charge:**

(a) Below is the orientation and coordinate system used to solve this problem. There are, of course, more ways than one to set this up:



(b) Following from this geometry:

$$V(\vec{r}) = \iint\limits_S \frac{\rho_S(\vec{r}')dS}{4\pi\epsilon||\vec{r}-\vec{r}'||} =$$

$$\underbrace{\frac{\rho_0}{4\pi\epsilon}\int_{d_1}^{d_1+d_0} dz' \int_0^L dy' \frac{1}{\sqrt{x^2+(y-y')^2+(z-z')^2}}}_{\text{Positive Charge Plate}} - \underbrace{\frac{\rho_0}{4\pi\epsilon}\int_0^{d_0} dz' \int_0^L dy' \frac{1}{\sqrt{(x-w_{GD})^2+(y-y')^2+(z-z')^2}}}_{\text{Negative Charge Plate}}$$

Looks messy, but now it is at least in a form that can be easily entered into a computer and evaluated.

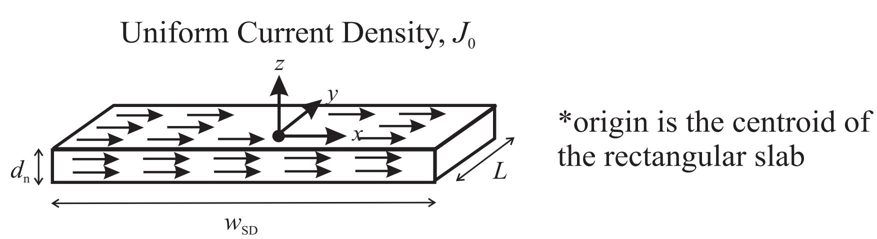
(c) If you know voltage as a function of 3D space, you can calculate field from the following relationship:

$$\vec{E}(x,y,z) = -\nabla V(x,y,z)$$

(3) **MOSFET Current:**

(a) A constant current density J_0 is flowing through a rectangular $L \times d_n$ area. Thus, $I = J_0 L d_n$.

(b) Below is the orientation and coordinate system used to solve this problem. There are, of course, more ways than one to set this up:



(c) Following from our geometry:

$$\vec{H}(\vec{r}) = \iiint\limits_V \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')dV}{4\pi||\vec{r}-\vec{r}'||^3} = \int_{-\frac{1}{2}w_{GD}}^{+\frac{1}{2}w_{GD}} dx' \int_{-\frac{L}{2}}^{+\frac{L}{2}} dy' \int_{-\frac{d_n}{2}}^{+\frac{d_n}{2}} dz' \frac{J_0\hat{x} \times [(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}]}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{3}{2}}}$$

This expression gets full credit. The intrepid may want to simplify further by distributing the cross-product:

$$\vec{H}(x,y,z) = J_0 \int_{-\frac{1}{2}w_{GD}}^{+\frac{1}{2}w_{GD}} dx' \int_{-\frac{L}{2}}^{+\frac{L}{2}} dy' \int_{-\frac{d_n}{2}}^{+\frac{d_n}{2}} dz' \frac{(y-y')\hat{z} - (z-z')\hat{y}}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{\frac{3}{2}}}$$

Now ready to use the computer!

(d) If this rectangular slab of current were the only current present, then the gate would be creating charges and sending them to the drain where they are instantly destroyed. In practice, of course, there are return current paths that carry charges away from the drain and back to the gate outside the MOSFET. Our solution, however, is not a bad approximation if we are studying H-fields in and around the MOSFET, where the rectangular slab of current is the dominant contributor.