ECE 3025: Electromagnetics

Solutions to TEST 3 (Fall 2006)

(1) **Laplace's Equation:**

(a) Taking the Laplacian of voltage in both regions of space:

$$\begin{aligned}
\nabla^2 V &= \begin{cases} \nabla^2 \frac{V_0\phi}{\alpha} & 0<\phi<\alpha \\ \nabla^2 \frac{V_0}{2\pi-\alpha}[2\pi-\phi] & \alpha<\phi<2\pi \end{cases} \\
&= \begin{cases} \frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left\{\frac{\partial}{\partial\phi}\frac{V_0\phi}{\alpha}\right\} & 0<\phi<\alpha \\ \frac{1}{\rho^2}\frac{\partial}{\partial\phi}\left\{\frac{\partial}{\partial\phi}\frac{V_0}{2\pi-\alpha}[2\pi-\phi]\right\} & \alpha<\phi<2\pi \end{cases} \\
&= \begin{cases} \frac{1}{\rho^2}\frac{\partial}{\partial\phi}\frac{V_0}{\alpha} & 0<\phi<\alpha \\ \frac{1}{\rho^2}\frac{\partial}{\partial\phi}\frac{-V_0}{2\pi-\alpha} & \alpha<\phi<2\pi \end{cases} \\
&= \begin{cases} 0 & 0<\phi<\alpha \\ 0 & \alpha<\phi<2\pi \end{cases}
\end{aligned}$$

which agrees with the chargeless region of space around the metal plates.

(b) Recognize that $\vec{E} = -\nabla V(\phi)$. The key difficulty in the problem is to make sure we use the gradient formula for the *cylindrical* coordinate system:

$$\begin{aligned}
-\nabla V(\phi) &= \begin{cases} -\nabla \frac{V_0\phi}{\alpha} & 0<\phi<\alpha \\ -\nabla \frac{V_0}{2\pi-\alpha}[2\pi-\phi] & \alpha<\phi<2\pi \end{cases} \\
&= \begin{cases} -\frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial\phi}\frac{V_0\phi}{\alpha} & 0<\phi<\alpha \\ -\frac{1}{\rho}\hat{\phi}\frac{\partial}{\partial\phi}\frac{V_0}{2\pi-\alpha}[2\pi-\phi] & \alpha<\phi<2\pi \end{cases} \\
&= \begin{cases} -\frac{V_0}{\rho\alpha}\hat{\phi} & 0<\phi<\alpha \\ \frac{V_0}{\rho(2\pi-\alpha)}\hat{\phi} & \alpha<\phi<2\pi \end{cases}
\end{aligned}$$

which is the stated solution for E-field.

(c) To find charge from an electric field distribution, we apply Gauss's law in differential form: $\nabla\cdot\vec{D} = \rho_v$. Note that the divergence formula in cylindrical coordinates predicts that

$$\nabla\cdot(\epsilon\vec{E}) = \epsilon\frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} = -\frac{\epsilon V_0}{\rho^2\alpha}\frac{\partial}{\partial\phi}\left[1-\frac{2\pi}{2\pi-\alpha}\mathrm{u}(\phi-\alpha)\right] = \frac{\epsilon V_0}{\rho^2\alpha}\frac{2\pi}{2\pi-\alpha}\delta(\phi-\alpha)$$

This result makes intuitive sense, confirming what we learned from part (a) that there is no charge in the region around the plates. For the region $\phi=\alpha$, however, the delta-function is non-zero, indicating the presence of surface charge on this plane. In fact, given the hint that $\rho_v(\vec{r}) = \rho_s(\rho,z)\frac{\delta(\phi-\alpha)}{\rho}$ (a result easily derived from the calculus of cylindrical coordinates) we can see that the top plate must contain the following surface charge distribution:

$$\rho_s(\rho,z) = +\frac{\epsilon V_0}{\rho\alpha}\left(\frac{2\pi}{2\pi-\alpha}\right)$$

which gradually tapers the otherwise uniform charge for locations away from the z-axis. The $\phi = 0$ plate would contain the negative counterpart of this charge.

Now we have enough information to estimate the per-unit-length capacitance. Calculating the charge per unit length along the z-axis:

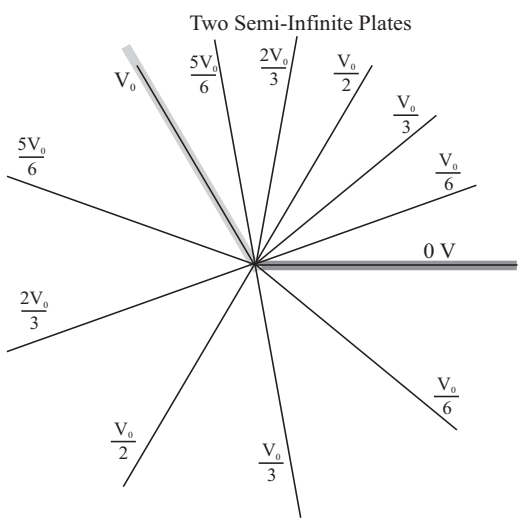
$$\rho_L = \int_a^b \rho_s \, d\rho = \frac{\epsilon V_0}{\alpha} \frac{2\pi}{2\pi - \alpha} \int_a^b \frac{d\rho}{\rho} = \frac{\epsilon V_0}{\alpha} \frac{2\pi}{2\pi - \alpha} \ln\left(\frac{b}{a}\right)$$

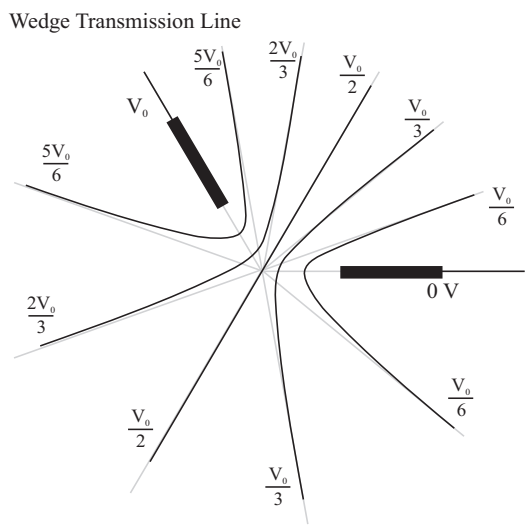
Plugging this result into our definition for per-unit-length capacitance produces

$$C = \frac{\rho_L}{V_0} = \frac{2\pi\epsilon}{\alpha(2\pi - \alpha)} \ln\left(\frac{b}{a}\right)$$

Does this answer make sense? Yes, because it exhibits all of the intuitive behavior we might expect of capacitance. As the surface area of the plates gets larger (the ration of b/a), capacitance increases. There is a local minimum for plates when they are bent farthest apart ($\alpha = \pi$); capacitance increases as the plates are brought closer together.

As mentioned in the problem statement, this field expression is somewhat approximate (it is exact only as $b \longrightarrow \infty$ and $a \longrightarrow 0$). Consider a cross-sectional sketch of equipotential lines for the case of the ideal semi-infinite wedge and the finite wedge:





Compared side by side, the solutions are almost identical except close-in to the z-axis. However, the solution is still an excellent approximation around the plates and useful for estimating surface charges.

(2) **Induction Charging:**

(a) The total magnetic flux through a circular coil of radius, R is given by

$$\Psi_M(t) = \iint_A \vec{B}(x, y, z, t) \cdot (dz\,\hat{z}) = \pi R^2 B_0 \sin(2\pi f_0 t)$$

According to Faraday's law, the time derivative of this magnetic flux is the voltage around the path, which must be multiplied by the N coil turns around this path:

$$V = N\frac{\partial \Psi_M}{\partial t} = 2\pi^2 f_0 R^2 N B_0 \cos(2\pi f_0 t)$$

(b) Acceptable answers include 1) increase current through the charging coil, 2) increase permeability of material inside coils, 3) increase the frequency of excitation, 4) increase the number of coil turns, N.

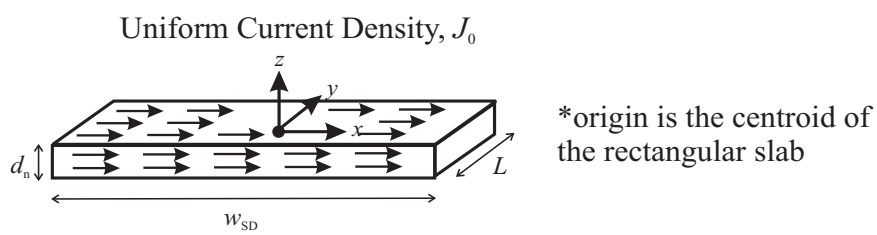
(c) Induction currents will form on any metallic car part within range of the magnetic field and, since those parts are not PECs, energy will be expended.

(d) Much like the RFID induction tag we studied in class and in homework, even modest increases in distance between the charging coil and the receiving coil of the vehicle will dramatically reduce the available power for coupling. Thus, the jacked-up pick-up truck should charge much more slowly than the small VW bug.

(3) **MOSFET Current:**

(a) A constant current density J_0 is flowing through a rectangular $L \times d_n$ area. Thus, $I = J_0 L d_n$.

(b) Below is the orientation and coordinate system used to solve this problem. There are, of course, more ways than one to set this up:



(c) Following from our geometry:

$$\vec{H}(\vec{r}) = \iiint_V \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')dV}{4\pi||\vec{r} - \vec{r}'||^3} = \int_{-\frac{1}{2}w_{GD}}^{+\frac{1}{2}w_{GD}} dx' \int_{-\frac{L}{2}}^{+\frac{L}{2}} dy' \int_{-\frac{d_n}{2}}^{+\frac{d_n}{2}} dz' \frac{J_0\hat{x} \times [(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}]}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{3}{2}}}$$

This expression gets full credit. The intrepid may want to simplify further by distributing the cross-product:

$$\vec{H}(x,y,z) = J_0 \int_{-\frac{1}{2}w_{GD}}^{+\frac{1}{2}w_{GD}} dx' \int_{-\frac{L}{2}}^{+\frac{L}{2}} dy' \int_{-\frac{d_n}{2}}^{+\frac{d_n}{2}} dz' \frac{(y-y')\hat{z} - (z-z')\hat{y}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{3}{2}}}$$

Now ready to use the computer!

(d) If this rectangular slab of current were the only current present, then the gate would be creating charges and sending them to the drain where they are instantly destroyed. In practice, of course, there are return current paths that carry charges away from the drain and back to the gate outside the MOSFET. Our solution, however, is not a

bad approximation if we are studying H-fields in and around the MOSFET, where the rectangular slab of current is the dominant contributor.