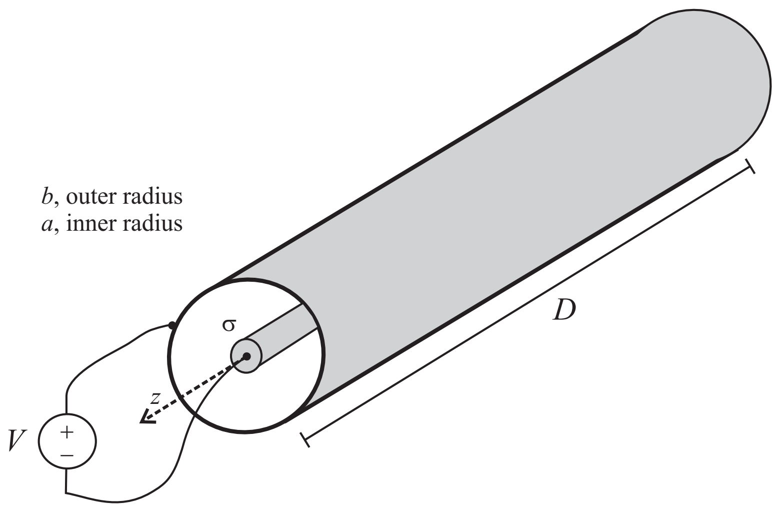
ECE 3025: Electromagnetics
Solutions to TEST 3 (Fall 2009)

1. **Coaxial Resistor:** A coaxial structure of length D is filled with a liquid with conductivity σ (Ω^{-1} m^{-1}) and held at voltage V by the source depicted below. The electric field inside the coaxial structure is given by

$$\vec{E}(\rho, \phi, z) = -\frac{K_0}{2\pi\rho\sigma}\hat{\rho}$$

where K_0 is a constant. Answer the following questions based on this scenario. (**50 points**)



(a) The units of K_0 are Amps/m.

(b) Current density flowing through this device is given by

$$\vec{J} = \sigma\vec{E} = -\frac{K_0}{2\pi\rho}\hat{\rho}$$

Integrating this over any z-axis centered cylindrical area such that $a \le \rho \le b$ gives the total current:

$$I = \iint_S \vec{J}(\vec{r}') \cdot d\hat{n} = \int_0^{2\pi}\int_0^{D} -\frac{K_0}{2\pi\rho}\hat{\rho} \cdot (-\rho \, d\phi \, dz \, hat\rho) = K_0 \, D$$

(c) Total resistance is defined as:

$$R = \frac{V}{I} = \frac{-\int \vec{E} \cdot d\vec{l}}{K_0 \, D} = \frac{1}{K_0 D}\int_a^b \frac{K_0}{2\pi\sigma\rho}\hat{\rho} \cdot d\rho \, \hat{\rho} = \frac{\ln(b/a)}{2\pi\sigma D}$$

(d) The total charge on the air-filled structure is

$$Q = CV = \frac{\epsilon_0 DV}{2\pi \ln(b/a)}$$

(e) It squirted out. Here is the most straight-forward explanation: the energy stored in the coaxial capacitor changes with and without material inside:

$$\underbrace{W = \frac{1}{2} C V^2}_{\text{with material } \epsilon} \qquad W = \underbrace{\frac{1}{2}}_{\text{with air } \epsilon_0} \mathrm{C}V^2$$

Since the voltage is the same in each case, the lower (preferred) energy state is an empty capacitive chamber. Interestingly, if the voltage was disconnected before the plug was pulled and the *charge* was the same, the opposite would happen (like the muscle example we studied in class):

$$\underbrace{W = \frac{1}{2}\frac{Q^2}{C}}_{\text{with material } \epsilon} \qquad W = \underbrace{\frac{1}{2}\frac{Q^2}{\mathrm{C}}}_{\text{with air } \epsilon_0}$$

2. **Simple Generator:** A square of length L would collect magnetic flux around its path as a function of angle θ with the xy-plane:

$$\Phi_M = B_0 L^2 \cos(\theta) \quad \text{where } \theta = 2\pi f t$$

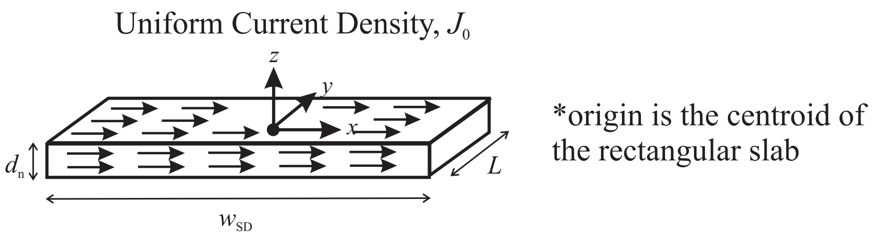
As it spins in the field, the voltage would be equal to the time derivative of this expression; the current would be equal to this voltage divided by the total resistance:

$$I(t) = -\frac{2\pi f B_0 L^2}{R}\sin(2\pi f t)$$

3. **MOSFET Current:**

(a) A constant current density J_0 is flowing through a rectangular $L \times d_n$ area. Thus, $I = J_0 L d_n$.

(b) Below is the orientation and coordinate system used to solve this problem. There are, of course, more ways than one to set this up:



(c) Following from our geometry:

$$\vec{H}(\vec{r}) = \iiint\limits_V \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')dV}{4\pi||\vec{r} - \vec{r}'||^3} = \int\limits_{-\frac{1}{2}w_{GD}}^{+\frac{1}{2}w_{GD}} dx' \int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} dy' \int\limits_{-\frac{d_n}{2}}^{+\frac{d_n}{2}} dz' \frac{J_0\hat{x} \times [(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}]}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{3}{2}}}$$

This expression gets full credit. The intrepid may want to simplify further by distributing the cross-product:

$$\vec{H}(x,y,z) = J_0 \int\limits_{-\frac{1}{2}w_{GD}}^{+\frac{1}{2}w_{GD}} dx' \int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} dy' \int\limits_{-\frac{d_n}{2}}^{+\frac{d_n}{2}} dz' \frac{(y-y')\hat{z} - (z-z')\hat{y}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{3}{2}}}$$

Now ready to use the computer!

(d) If this rectangular slab of current were the only current present, then the gate would be creating charges and sending them to the drain where they are instantly destroyed. In practice, of course, there are return current paths that carry charges away from the drain and back to the gate outside the MOSFET. Our solution, however, is not a bad approximation if we are studying H-fields in and around the MOSFET, where the rectangular slab of current is the dominant contributor.