ECE4370 HW2 Solution

Huan Yu

Problem 1

The vector magnetic potential for a Hertzian dipole is given by

$$\tilde{\vec{\mathrm{A}}} = \frac{\mu I\,dl}{4\pi r}\exp(-jkr)\,\hat{\mathrm{z}}$$

$$\begin{aligned}
(\nabla^2 + k^2)\tilde{\vec{\mathrm{A}}} &= \frac{\mu I\,dl}{4\pi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{\exp(-jkr)}{r}\right) + k^2\frac{\exp(-jkr)}{r}\right]\hat{\mathrm{z}} \\
&= \frac{\mu I\,dl}{4\pi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(-jkr\exp(-jkr) - \exp(-jkr)\right) + k^2\frac{\exp(-jkr)}{r}\right]\hat{\mathrm{z}} \\
&= \frac{\mu I\,dl}{4\pi}\left[-jk\frac{\exp(-jkr)}{r^2} - k^2\frac{\exp(-jkr)}{r} + jk\frac{\exp(-jkr)}{r^2} + k^2\frac{\exp(-jkr)}{r}\right]\hat{\mathrm{z}} \\
&= 0\,\text{for } r > 0,\ \infty/\text{undefined at } r = 0
\end{aligned}$$

This result is completely consistent with the impulsive current density source of a Hertzian dipole – 0 everywhere except the origin, where the current density is impulsive.

Problem 2

$$\begin{aligned}
\vec{A}_z(\vec{r}) &= \frac{\mu I\,dl}{4\pi}\frac{e^{-jkr}}{r}\hat{z} \\
&= \frac{\mu I\,dl}{4\pi}\frac{e^{-jkr}}{r}\cos\theta\hat{r} - \frac{\mu I\,dl}{4\pi}\frac{e^{-jkr}}{r}\sin\theta\hat{\theta}
\end{aligned}$$

$$\begin{aligned}\vec{H}(\vec{r}) &= \frac{\bigtriangledown \times \vec{A}_z(\vec{r})}{\mu} \\ &= \frac{1}{\mu}(-\frac{1}{r}\hat{\phi}\frac{\partial A_r}{\partial \theta} + \frac{1}{r}\hat{\phi}\frac{\partial(rA_\theta)}{\partial r}) \\ &= \frac{jIdlk\sin\theta}{4\pi r}e^{-jkr}[1-\frac{j}{kr}]\hat{\phi}\end{aligned}$$

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{\bigtriangledown \times \vec{H}(\vec{r})}{j2\pi f\epsilon} \\ &= \frac{1}{j2\pi f\epsilon}[\frac{1}{r^2\sin\theta}\hat{r}\frac{\partial(r\sin\theta H_\phi)}{\partial\theta} - \frac{1}{r\sin\theta}\hat{\theta}\frac{\partial(r\sin\theta H_\phi)}{\partial r}] \\ &= \frac{jIdlk\eta\sin\theta}{4\pi r}e^{-jkr}[1-\frac{j}{kr}-\frac{1}{(kr)^2}]\hat{\theta} + \frac{jIdlk\eta\sin\theta}{4\pi r}e^{-jkr}[-j\frac{2\cot\theta}{kr} - \frac{2\cot\theta}{(kr)^2}]\hat{r}\end{aligned}$$

Problem 3

$$\begin{aligned}|\vec{A}_z(\vec{r})| &= \frac{\mu}{4\pi r}|\int_{-\frac{L}{2}}^{\frac{L}{2}} I_0(1-\frac{2|z|}{L})\exp(jkz'cos\theta)dz'| \\ &= \frac{\mu I_0}{2\pi r}|\int_0^{\frac{L}{2}}(1-\frac{2|z|}{L})\cos(\frac{2\pi}{\lambda}z'\cos\theta)dz'|\end{aligned}$$

For $L \ll \lambda$

$$\begin{aligned}|\vec{A}_z(\vec{r})| &\approx \frac{\mu I_0}{2\pi r}\int_0^{\frac{L}{2}}(1-\frac{2|z|}{L})dz' \\ &= \frac{\mu I_0 L}{8\pi r}\end{aligned}$$

$$\begin{aligned}PowerDensity(W/m^2): &= \frac{k^2\eta\sin^2\theta}{2\mu^2}|\vec{A}_z|^2 \\ &= \frac{\eta I_0^2}{r^2}\frac{L^2}{32\lambda^2}\sin^2\theta\end{aligned}$$

Note that this is the same basic result as the Hertzian dipole, differing only by a constant of proportionality. This leads to similar radiation parameters.
The radiation resistance$=20\pi^2(\frac{L}{\lambda})^2$