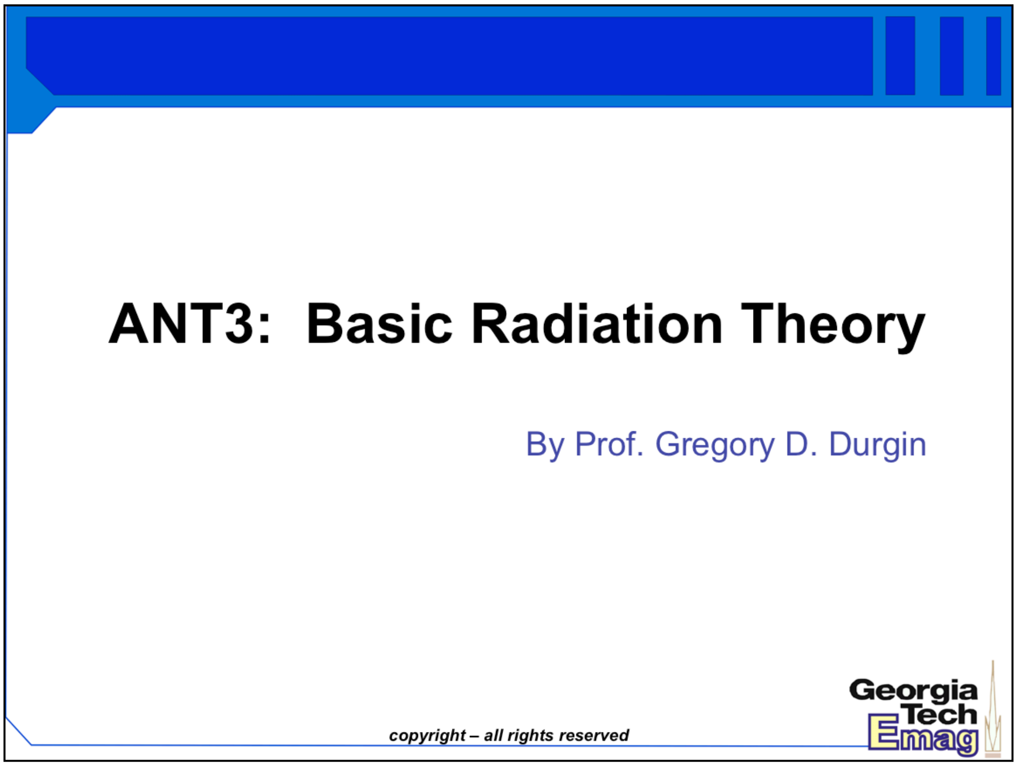


In this module, we discuss the basic approach for solving electromagnetic field quantities given a distribution of current in space.

Radiation from Impressed Currents

The Helmholtz wave equation presumes a simple, source-free medium. But if we have an antenna, we need to study radiation patterns from current distributions, i.e. sources:

$$\nabla \times \vec{\tilde{\mathrm{H}}} = j2\pi f \epsilon \vec{\tilde{\mathrm{E}}} + \vec{\tilde{\mathrm{J}}}$$

$$\text{Current Distribution: } \vec{\tilde{\mathrm{J}}}(\vec{\mathrm{r}})$$

$$\nabla \times \vec{\tilde{\mathrm{E}}} = -j2\pi f \mu \vec{\tilde{\mathrm{H}}}$$

Two types of radiation problems to solve analytically:

- Known current distribution on antenna
- Known field distribution around antenna

Georgia Tech Emag

In a classical electromagnetic description of waves, both electric and magnetic field components vary as a function of three-dimensional space according to the Helmholtz (scalar) wave equation. This relationship holds for simple media (homogenous, isotropic, linear) that are source free (no currents or charges) under time-harmonic excitation. However, an active antenna element must introduce current into the system, violating the source-free assumption of the Helmholtz wave equation. Thus, we will need an alternative approach to solving a radiating system with currents.

The approach discussed in the next section assumes that we know current a priori and will attempt to find the electromagnetic fields from this given quantity. In a way, though, this is a chicken-and-egg problem: the current distribution on an antenna affects the field distribution in space and the field distribution in space affects the current distribution on an antenna. There are a family of antennas for which the current distribution on metal is relatively easy to estimate a priori, so this approach is a good place to start. As we build more sophistication into our modeling efforts, we will have to solve *integral equations* to get around the chicken-and-egg problem.

Introduction of Vector Magnetic Potential

$$\text{Scalar Electric Potential: } \vec{\mathrm{E}} = -\nabla\tilde{\Phi}_e$$

$$\text{Vector Magnetic Potential: } \tilde{\vec{\mathrm{H}}} = \frac{1}{\mu}\nabla\times\tilde{\vec{\mathrm{A}}}$$

Thus, electric field relates to vector magnetic potential

$$\nabla\times\tilde{\vec{\mathrm{E}}} = -j2\pi f\mu\tilde{\vec{\mathrm{H}}} = -j2\pi f\nabla\times\tilde{\vec{\mathrm{A}}}$$

Therefore,

$$\tilde{\vec{\mathrm{E}}} = -j2\pi f\tilde{\vec{\mathrm{A}}} - \nabla\tilde{\Phi}_e$$

Georgia Tech Emag

We will introduce a new quantity which *does* solve the scalar wave equation when a medium has currents present. This quantity, represented by an A-vector, is called the vector magnetic potential. It functions similar to the scalar electric potential (voltage function) in that magnetic field can be derived from this quantity by taking the curl and dividing by the permeability. The A-vector was actually first introduced by Maxwell himself in his original treatise on electromagnetic waves and was considered a derived quantity. There are some physicists, however, who believe it to be more fundamental, that *all* of electromagnetic theory can be described in terms of the vector magnetic and scalar electric potentials. Carver Mead, a retired Cal Tech physicist and colleague of Richard Feynman, wrote a text that explains how this quantity allows us to solve some problems easier than the classical E/H formulation. Physicists like this approach because it is possible to describe forces on charges entirely without direct dealings with H/B-fields, which physicists see as unnecessary since magnetism is really just electric attraction with the effects of relativity taken into consideration. The A-field goes by the names vector magnetic potential, retarded potential (older texts, largely due to its -90 degree lag of E), or electrodynamic momentum.

Development of a Scalar Wave Equation

Substitute for electric field:

$$\frac{1}{\mu}\nabla \times \nabla \times \tilde{\vec{A}} = j2\pi f\epsilon\left[-j2\pi f\tilde{\vec{A}} - \nabla\tilde{\Phi}_e\right] + \tilde{\vec{J}}$$

Apply identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot A) - \nabla^2\vec{A}$:

$$\nabla \times \nabla \times \tilde{\vec{A}} = 4\pi^2 f^2 \mu\epsilon\tilde{\vec{A}} - j2\pi f\epsilon\mu\nabla\tilde{\Phi}_e + \tilde{\vec{J}}$$

$$\left[\nabla^2 + k^2\right]\tilde{\vec{A}} - \underbrace{\nabla\left(\nabla \cdot \tilde{\vec{A}} + j2\pi f\mu\epsilon\tilde{\Phi}_e\right)}_{\text{apply gauge condition}} = -\mu\tilde{\vec{J}}$$

When we insert the vector magnetic potential into Maxwell's equations, we arrive at something that looks a lot like the Helmholtz/scalar wave equation. Two key differences: 1) there is now a source term (current density, J-vector) on the right-hand side and 2) there is an extra term that complicates the solution. This term is not a problem though, because we have not fully specified the A-vector – only its curl. To fully specify a vector field, we must describe its curl and its divergence. Thus, if we pick a clever choice of divergence for A, we can make this second term disappear. This choice of divergence for vector A is called the Lorentz Gage.

Scalar Wave Equation

Now we have a much simpler equation for solving $\vec{A}$. This is actually a *scalar* wave equation like the Helmholtz wave equation, since the x-, y-, and z-components of $\tilde{\vec{A}}$ are described by 3 separable scalar equations.

$$\left[\nabla^2 + k^2\right] \tilde{\vec{A}} = -\mu \tilde{\vec{J}}$$

Our methodology is to solve for $\tilde{\vec{A}}$ and then calculate electric and magnetic field components from this vector quantity.

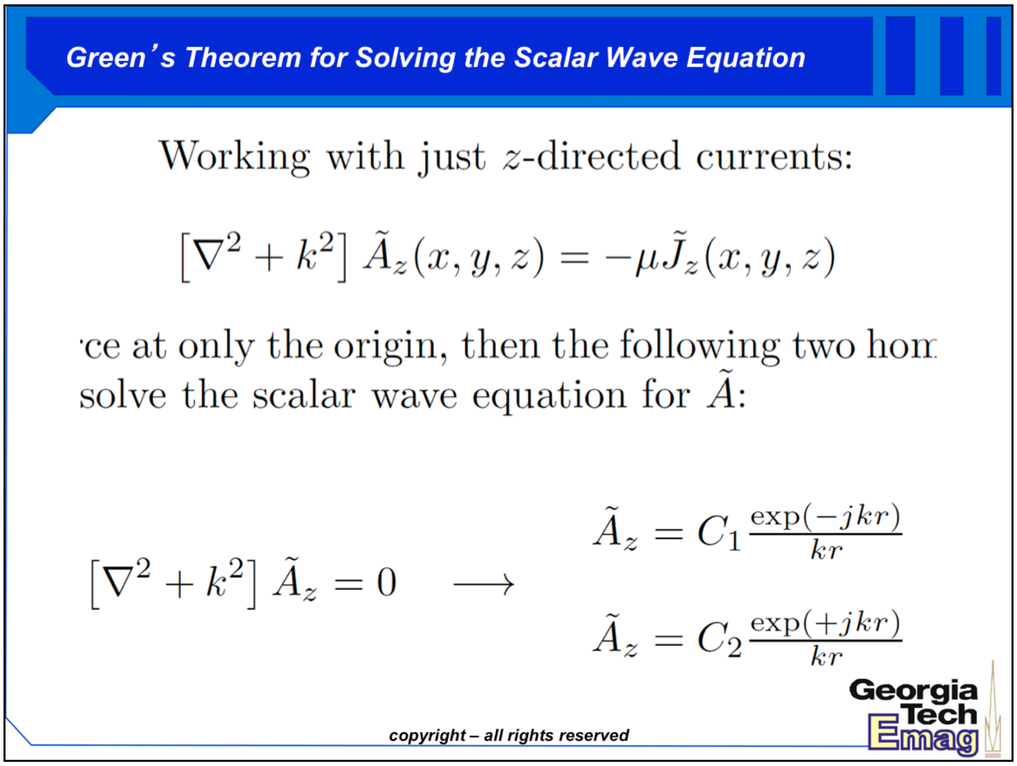
$$\tilde{\vec{H}} = \frac{1}{\mu} \nabla \times \tilde{\vec{A}}$$

$$\tilde{\vec{E}} = \frac{1}{j2\pi f \epsilon} \nabla \times \tilde{\vec{H}} = \frac{1}{j2\pi f \epsilon \mu} \nabla \times \nabla \times \tilde{\vec{A}}$$

Now we have a clean scalar wave equation for vector magnetic potential.

Our method for solving antenna fields given a radiating current then follows naturally:

1) We will solve for A-vector given a current density vector J
2) Once we have the A-vector, we can solve for the classical magnetic field H
3) Once we have the H-vector, we can solve for the classical electric field E
4) We can use classical transport equations and the Poynting vector to figure out where waves are traveling in space and how much power they carry



We will start with a simplified version of this analysis, considering a z-directed current element in free space. Since the wave equation is scalar, this implies a z-directed scalar Az as the only non-zero vector component in such a system.

There are two homogeneous solutions for the function Az in space if the source term is removed from everywhere except the origin. Once is a wave propagating away from the origin, the other is a wave that collapses to the origin from infinity. The latter term makes mathematical sense, but is physically meaningless, so we will drop it.

Green's Theorem

Green's Theorem applied to vector magnetic pot:

$$\tilde{A}_z(x,y,z) = \frac{\mu}{4\pi} \int_{\mathcal{V}} \tilde{J}_z(x',y',z') \, \frac{\exp(-jkR)}{R} \, dv'$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

point of observation: $\vec{\mathrm{r}} = x\hat{\mathrm{x}} + y\hat{\mathrm{y}} + z\hat{\mathrm{z}}$

variables of integration: $\vec{\mathrm{r}}' = x'\hat{\mathrm{x}} + y'\hat{\mathrm{y}} + z'\hat{\mathrm{z}}$

$$\tilde{A}_z(x,y,z) = \frac{\mu}{4\pi} \iiint_{\mathcal{V}} \tilde{J}_z(x',y',z') \, \frac{\exp\left(-jk \left\|\vec{\mathrm{r}}' - \vec{\mathrm{r}}\right\|\right)}{\left\|\vec{\mathrm{r}}' - \vec{\mathrm{r}}\right\|} \, dx' \, dy' \, dz'$$

This simple solution for Az given a possible source at the origin can be used to synthesize a field from any z-directed spatial distribution of current. Thus, plug any given Jz into the equation above and solve for Az. Recall that the standard quantities of electromagnetics – E-field, H-field, and Power – may all be derived from Az after this integration is complete.