ANT1: Basic Radiation Theory

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Radiation from Impressed Currents

Helmholtz presumes free-space, source-free medium. But if we want to study radiation patterns from current distributions, we need sources.

$\tilde{\vec{J}}(\vec{r})$

$$\nabla \times \tilde{\vec{H}} = j\omega\epsilon \tilde{\vec{E}} + \tilde{\vec{J}}$$ impressed current

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}}$$

Two types of Antenna problems

- Know the currents a priori, solve for radiation
- need to solve for radiation and current distribution (chicken and egg problem).

Introduction of Vector Magnetic Potential

Radiation from Impressed Currents

Electric Potential

$$\tilde{\vec{E}} = -\nabla \tilde{\Phi}_e$$

(vector) Magnetic Potential

$$\tilde{\vec{H}} = \frac{1}{\mu} \nabla \times \tilde{\vec{A}}$$

Thus

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}} = -j\omega \nabla \times \tilde{\vec{A}}$$

implies

$$\tilde{\vec{E}} = -j\omega \tilde{\vec{A}} - \nabla \tilde{\Phi}_e$$

this one always curls to zero

Development of a Scalar Wave Equation

Now

$$\nabla \times \tilde{\vec{H}} = \frac{1}{\mu} \nabla \times \nabla \times \tilde{\vec{A}} = j\omega\epsilon \tilde{\vec{E}} + \tilde{\vec{J}}$$

$$\nabla \times \nabla \times \tilde{\vec{A}} = j\omega\epsilon\mu \left[-j\omega \tilde{\vec{A}} - \nabla \tilde{\Phi}_e \right] + \mu \tilde{\vec{J}}$$

$$\nabla(\nabla \cdot \tilde{\vec{A}}) - \nabla^2 \tilde{\vec{A}} = +\underbrace{\omega^2 \epsilon\mu}_{k^2} \tilde{\vec{A}} - j\omega\epsilon\mu \nabla \tilde{\Phi}_e + \mu \tilde{\vec{J}}$$

$$(\nabla^2 + k^2)\tilde{\vec{A}} - \nabla\Big(\underbrace{\nabla \cdot \tilde{\vec{A}} + j\omega\epsilon\mu \tilde{\Phi}_e}_{\text{still have freedom to choose/ specify } \nabla \cdot \tilde{\vec{A}} = -j\omega\epsilon\mu \tilde{\Phi}_e}\Big) = -\mu \tilde{\vec{J}}$$

Scalar Wave Equation

Now

$$(\nabla^2 + k^2)\tilde{\vec{A}} = -\mu \tilde{\vec{J}}$$

Much more solvable. So once we get $\tilde{\vec{A}}$,

$$\tilde{\vec{H}}(\vec{r}) = \frac{1}{\mu} \nabla \times \tilde{\vec{A}}$$ and

$$\tilde{\vec{E}}(\vec{r}) = \frac{1}{j\omega\epsilon} \nabla \times \tilde{\vec{H}} = \frac{1}{j\omega\mu\epsilon} \nabla \times \nabla \times \tilde{\vec{A}}$$

Green's Theorem for Solving the Scalar Wave Equation

Let's assume z-oriented currents

$$(\nabla^2 + k^2)\tilde{A}_z \hat{z} = -\frac{1}{\mu} J_z(x,y,z)\hat{z}$$

Homogeneous Solution (origin exclusion)

$$(\nabla^2 + k^2)\tilde{A}_z = 0 \qquad \tilde{A}_{z1} = C_1 \frac{\exp(-jkr)}{r}$$

non-physical "collapsing power" $\rightarrow$ $\tilde{A}_{z2} = C_2 \frac{\exp(+jkr)}{r}$

Example: Ideal (Hertzian) Dipole

Green's theorem

$$\tilde{A}_z(x,y,z) = \frac{\mu}{4\pi}\iiint \tilde{J}_z(x',y',z')\frac{\exp(-jkR)}{R}\,dx'dy'dz'$$

$$R=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

Hertzian Dipole

$$\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$$

$$\tilde{J}_z(\vec{r}) = I\,dl\,\delta(x)\,\delta(y)\,\delta(z) = I\,\delta(\vec{r})$$ small current element on the origin

$$\tilde{A}_z(x,y,z) = \frac{\mu I\,dl}{4\pi}\hat{z}\,\frac{\exp\left(-jk\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\mu I\,dl}{4\pi}\hat{z}\,\frac{\exp(-jkr)}{r}$$

Solution for the Ideal Dipole

Fields from Hertzian Dipole

$$\tilde{\vec{H}}(\vec{r}) = \frac{1}{\mu}\nabla\times\left(\tilde{A}_z\hat{z}\right) = j\frac{I\,d\ell\,k\sin\theta}{4\pi r}\left(1+\frac{1}{jkr}\right)\exp(-jkr)\hat{\phi}$$

$$\tilde{\vec{E}}(\vec{r}) = \frac{\nabla\times\nabla\times\left(\tilde{A}_z\hat{z}\right)}{j\omega\mu\epsilon} =$$

$$\exp(-jkr)\left[k\frac{\eta I\cos\theta\,d\ell}{2\pi r^2}\left(1+\frac{1}{jkr}\right)\hat{r} + kj\frac{\eta I\,d\ell\sin\theta}{4\pi r}\left(1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right)\hat{\theta}\right]$$

Usually interested in far field $kr >> 1$ $(r >> \lambda)$

$$\tilde{\vec{H}}(\vec{r}) \cong j\frac{I\,d\ell\,k\sin\theta}{4\pi r}\exp(-jkr)\hat{\phi}$$

$$\tilde{\vec{E}}(\vec{r}) \cong j\eta\frac{I\,d\ell\,k\sin\theta}{4\pi r}\exp(-jkr)\hat{\theta}$$

Ideal Dipole: Total Radiated Power

Usually interested in Power

$$\tilde{\vec{S}}(r,\theta,\phi) = \frac{1}{2}\mathrm{Re}\left\{\tilde{\vec{E}} \times \tilde{\vec{H}}^*\right\} = \frac{I^2 dl^2 k^2 \sin^2\theta\, \eta}{2(4\pi r)^2}\,\hat{r}$$

Poynting Vector

Total Power

$$P_T = \iint_A d\hat{n} \cdot \vec{S} = \int_0^{\pi}\int_0^{2\pi} \underbrace{r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}}_{d\hat{n}} \cdot \frac{I^2 dl^2 k^2 \sin^2\theta\, \eta}{32\pi^2 r^2}\,\hat{r}$$

$$= \frac{I^2 dl^2 \eta (2\pi/\lambda)^2}{16\pi} \int_0^{\pi} \sin^3\theta\, d\theta$$

Look-up

$$\int_0^{\pi} \sin^3\theta\, d\theta = \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]\Big|_0^{\pi}$$

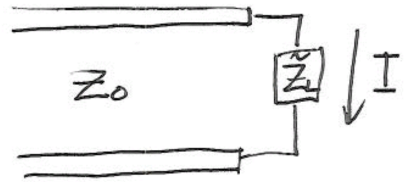
$$= \left[-(-1) - \frac{1}{3}\right] - \left[-1 + \frac{1}{3}\right] = \frac{4}{3}$$

Total Power

$$P_T = \frac{I^2 dl^2 \eta \pi}{3\lambda^2}$$

Ideal Dipole: Characteristic Impedance

Characteristic Impedance of Hertzian/Ideal Dipole



$$\tilde{Z}_L = R_A + jX_A$$

real radiation impedance

very large for small $d\ell$

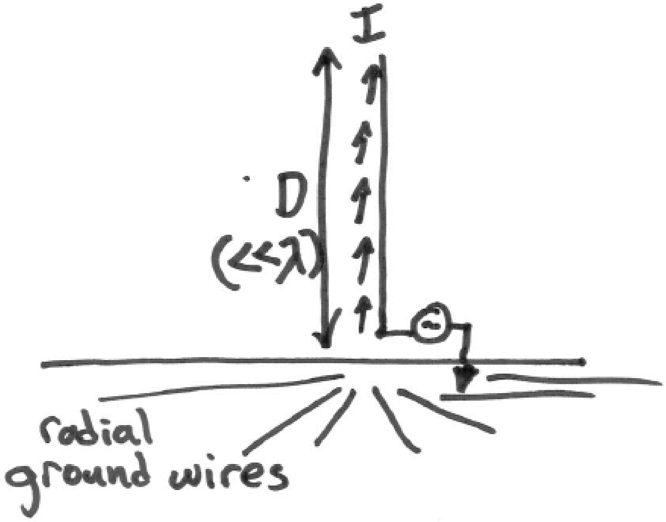
$$P_T = \frac{1}{2}\cancel{I^2} R_A = \frac{\cancel{I^2}\, d\ell^2 \eta \pi}{3\lambda^2} \quad \therefore \quad R_A = \frac{2\pi\eta}{3}\left(\frac{d\ell}{\lambda}\right)^2$$

$$X_A \cong -j Z_o \cot \frac{2\pi}{\lambda} d\ell$$ very capacitive

$$\boxed{\tilde{Z}_L \cong \frac{2\pi\eta}{3}\left(\frac{d\ell}{\lambda}\right)^2 - j Z_o \cot \frac{2\pi d\ell}{\lambda}}$$

Mismatch is serious for short antennas

Examples of Short Dipoles and Mismatch Problems



Directivity and Gain

$$D(\theta,\phi) = \frac{(4\pi r^2)\,\|\vec{S}(r,\phi,\theta)\|}{P_T}$$ ← total radiated power

$$= \frac{(\cancel{4}\cancel{\pi}\cancel{r^2})\,\dfrac{\cancel{I^2}\,\cancel{dl^2}\,k^2 \sin^2\theta\,\cancel{\eta}}{32\pi^2\cancel{r^2}}}{\cancel{I^2}\,\cancel{dl^2}\,\cancel{\eta}\,\cancel{\pi}/3\lambda^2} = \frac{12\lambda^2\left(2\pi/\lambda\right)^2 \sin^2\theta}{32\pi^2}$$

$$= \frac{3}{2}\sin^2\theta$$

$$\text{Gain} = D(\theta,\phi) \times \eta$$

efficiency (100% for PEC, no dielectric losses)

Half-Power Beamwidth

$$\text{Max Gain: } 10\log_{10}\left(D(\theta=\pi/2,\phi)\times 100\%\right) = 1.8\,\text{dBi}$$
$$(3/2 \text{ linear})$$

Half-Power Beamwidth

$$G(\theta'_{HPBW},\phi) = \frac{1}{2}G(\theta_{max}=\pi/2,\phi)$$

$$\frac{3}{2}\sin^2\theta'_{HPBW} = \frac{3}{4} \Rightarrow \theta_{HPBW1} = \pi/4$$
$$\theta_{HPBW2} = 3\pi/4$$

$$\theta_{HPBW} = \theta_{HPBW2} - \theta_{HPBW1} = 90^\circ$$