ECE 6390 Homework 3: Waves and Link Budgets

Solutions

Problem 1: Wave Equation

Show that the plane wave and spherical wave solution forms for electric field solve the Helmholtz wave equation in a simple, source-free medium. Are there any other assumptions that must be made? (5 points)

Solution

The Helmholtz equation is:

$$(\nabla^2 + k^2)\tilde{\vec{E}} = 0 \qquad \Rightarrow \qquad \nabla^2\tilde{\vec{E}} = -k^2\tilde{\vec{E}}$$

The time independent, phasor form of a plane wave is:

$$\tilde{\vec{E}} = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-j(x\vec{k}\cdot\hat{x}+y\vec{k}\cdot\hat{y}+z\vec{k}\cdot\hat{z})}$$

The vector Laplacian in Cartesian coordinates is calculated by taking the scalar Laplacian of each coordinate:

$$\nabla^2\tilde{\vec{E}} = \hat{x}\nabla^2(\tilde{\vec{E}}\cdot\hat{x}) + \hat{y}\nabla^2(\tilde{\vec{E}}\cdot\hat{y}) + \hat{z}\nabla^2(\tilde{\vec{E}}\cdot\hat{z})$$

Calculate the Laplacian for one coordinate ($\hat{i} = \hat{x}, \hat{y}, \hat{z}$)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\vec{E}_0\cdot\hat{i})e^{-j(x\vec{k}\cdot\hat{x}+y\vec{k}\cdot\hat{y}+z\vec{k}\cdot\hat{z})} =$$

$$-\left[(\vec{k}\cdot\hat{x})^2 + (\vec{k}\cdot\hat{y})^2 + (\vec{k}\cdot\hat{z})^2\right](\vec{E}_0\cdot\hat{i})e^{-j(x\vec{k}\cdot\hat{x}+y\vec{k}\cdot\hat{y}+z\vec{k}\cdot\hat{z})}$$

So the total vector Laplacian is:

$$\nabla^2\tilde{\vec{E}} = \underbrace{-\left[(\vec{k}\cdot\hat{x})^2 + (\vec{k}\cdot\hat{y})^2 + (\vec{k}\cdot\hat{z})^2\right]}_{-||\vec{k}||^2}\underbrace{\left[(\vec{E}_0\cdot\hat{x})\hat{x} + (\vec{E}_0\cdot\hat{y})\hat{y} + (\vec{E}_0\cdot\hat{z})\hat{z}\right]}_{\vec{E}_0} e^{-j(x\vec{k}\cdot\hat{x}+y\vec{k}\cdot\hat{y}+z\vec{k}\cdot\hat{z})}$$

$$= -||\vec{k}||^2\tilde{\vec{E}}$$

So the particular plane wave we considered satisfies the Helmholtz equation when magnitude of the wavevector (the wavenumber) matches the constant k from the equation (when $||\vec{k}|| = k$). Now consider the form of a spherical wave:

$$\tilde{\vec{E}} = E_0(\phi,\theta)\frac{e^{-jkr}}{(r/\lambda)}\hat{e} = E_0(\phi,\theta)\frac{\alpha e^{-jkr}}{(r/\lambda)}\hat{\phi} + E_0(\phi,\theta)\frac{\beta e^{-jkr}}{(r/\lambda)}\hat{\theta}$$

Note that we've taken the direction of the field vector as having arbitrary ϕ and θ components, i.e. $\hat{e} = \alpha\hat{\phi} + \beta\hat{\theta}$. The vector Laplacian in spherical coordinates is rather complex. Component by component, it is:

$$\hat{r} \cdot \nabla^2 \tilde{\vec{E}} = \nabla^2 E_r - \frac{2E_r}{r^2} - \frac{2\cot\theta}{r^2} E_\theta - \frac{2}{r^2}\frac{\partial E_\theta}{\partial \theta} - \frac{2}{r^2 \sin\theta}\frac{\partial E_\phi}{\partial \phi} \tag{1}$$

$$\hat{\theta} \cdot \nabla^2 \tilde{\vec{E}} = \nabla^2 E_\theta - \frac{2}{r^2}\frac{\partial E_r}{\partial \theta} - \frac{E_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\phi}{\partial \phi} \tag{2}$$

$$\hat{\phi} \cdot \nabla^2 \tilde{\vec{E}} = \nabla^2 E_\phi + \frac{2}{r^2\sin^2\theta}\frac{\partial E_r}{\partial \phi} - \frac{E_\phi}{r^2\sin^2\theta} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial E_\theta}{\partial \phi} \tag{3}$$

We'll need the scalar Laplacian operator expressed in spherical coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}$$

We'll start with a large simplification; we will neglect any terms that will have an r dependence which drops off faster than $1/r$. For our case, $E_r = 0$ as we can see it has no $\hat{r}$ component. Using these two conditions in (1), its clear that see that $\hat{r} \cdot \nabla^2 \tilde{\vec{E}} \approx 0$. As for (2) and (3), we'll similarly neglect all the terms except the one involving the scalar Laplacian, since those all fall off as $1/r^3$. So:

$$\hat{\theta} \cdot \nabla^2 \tilde{\vec{E}} \approx \nabla^2 E_\theta = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2}\right] E_0(\phi,\theta)\frac{\beta e^{-jkr}}{(r/\lambda)}$$

Now again, neglect the terms with $1/r^2$ dependence, giving the simplification:

$$\hat{\theta} \cdot \nabla^2 \tilde{\vec{E}} \approx \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\right] E_0(\phi,\theta)\frac{\beta e^{-jkr}}{(r/\lambda)}$$

$$= E_0(\phi,\theta)\frac{\beta}{1/\lambda}\frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{e^{-jkr}(-jk)r - e^{-jkr}}{\cancel{r^2}}\right)$$

$$= E_0(\phi,\theta)\frac{\beta}{1/\lambda}\frac{1}{r^2}\Big[(-jk)\left(e^{-jkr} + re^{-jkr}(-jk)\right) - e^{-jkr}(-jk)\Big] =$$

$$E_0(\phi,\theta)\frac{\beta}{1/\lambda}\left(\frac{e^{-jkr}(-jk)}{r^2} + \frac{-k^2\cancel{r}e^{-jkr}}{r^{\cancel{2}}} - \frac{e^{-jkr}(-jk)}{r^2}\right)$$

Finally, neglecting the $1/r^2$ terms again, we have:

$$\hat{\theta} \cdot \nabla^2 \tilde{\vec{E}} \approx -k^2 E_0(\phi,\theta)\frac{\beta e^{-jkr}}{r/\lambda} = -k^2 E_\theta$$

The same series of approximations, and the exact same calculations, yeild the same result for the $\hat{\phi}$ component:

$$\hat{\phi} \cdot \nabla^2 \tilde{\vec{E}} \approx \nabla^2 E_\phi \approx \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)\right] E_\phi \approx -k^2 E_\phi$$

The two components, $\hat{\phi}$ and $\hat{\theta}$ togehter show that $\nabla^2 \tilde{\vec{E}} \approx -k^2 \tilde{\vec{E}}$ for this spherical wave.

Problem 2: Plane Wave

Below is the electric field solution for a free-space plane wave:

$$\tilde{\vec{E}}(\vec{r}) = 10\,[3\hat{x} - 4\hat{y} + 5\hat{z}]\exp(-j[2\hat{x} - \hat{y} - 2\hat{z}] \cdot \vec{r}) \text{ mV/m}$$

(a) What is the corresponding $\tilde{\vec{H}}(\vec{r})$ that accompanies this electric field? What is the wavelength of this plane wave? (5 points)

(b) If $\hat{z}$ corresponds to the vertical direction, $\hat{x}$ corresponds to due east along the horizon, and $\hat{y}$ corresponds to due north along the horizon, what look angle (azimuth and elevation) should you use to point a dish antenna for receiving this wave? (5 points)

Solution, part (a)

First recognize that the field is of this form:

$$\tilde{\vec{E}}(\vec{r}) = E_0 \hat{e} \exp(-j(\phi_0 - k\hat{k} \cdot \vec{r}))$$

And the magnetic field for a plane wave will have this this form:

$$\tilde{\vec{H}}(\vec{r}) = \frac{E_0}{\eta}(\hat{k} \times \hat{e}) \exp(-j(\phi_0 - k\hat{k} \cdot \vec{r})) \tag{4}$$

From the given plane wave, we identify the parameters:

$$E_0 = 10\sqrt{3^2 + 4^2 + 5^2} = 50\sqrt{2} \qquad \hat{e} = \frac{3\hat{x} - 4\hat{y} + 5\hat{z}}{5\sqrt{2}} \qquad \phi_0 = 0$$

$$k = \sqrt{2^2 + 1^2 + 2^2} = 3 \qquad \hat{k} = \frac{2\hat{x} - \hat{y} - 2\hat{z}}{3}$$

Now we can calculate the wavelength:

$$\boxed{\lambda = \frac{2\pi}{k} = \frac{2\pi}{3} \text{ m}}$$

Now calculate the direction:

$$\hat{k} \times \hat{e} = \frac{2\hat{x} - \hat{y} - 2\hat{z}}{3} \times \frac{3\hat{x} - 4\hat{y} + 5\hat{z}}{5\sqrt{2}} = \frac{1}{15\sqrt{2}}(-8\hat{z} - 10\hat{y} + 3\hat{z} - 5\hat{x} - 6\hat{y} - 8\hat{x}) = \frac{-13\hat{x} - 16\hat{y} - 5\hat{z}}{15\sqrt{2}}$$

And plug this along with the other known quantities into (4):

$$\tilde{\vec{H}}(\vec{r}) = \frac{50\sqrt{2}}{\eta} \frac{-13\hat{x} - 16\hat{y} - 5\hat{z}}{15\sqrt{2}} \exp(-j[2\hat{x} - \hat{y} - 2\hat{z}] \cdot \vec{r})$$

Simplifying and plugging in the free space impedance $\eta \approx 120\pi$:

$$\boxed{\tilde{\vec{H}}(\vec{r}) = \frac{1}{36\pi}(-13\hat{x} - 16\hat{y} - 5\hat{z}) \exp(-j[2\hat{x} - \hat{y} - 2\hat{z}] \cdot \vec{r}) \text{ mA/m}}$$

Solution, part (b)

To receive this wave, the dish vector (the direction in which it points) should be the negative of the wavevector so that the dish faces the incident wave:

$$\hat{r} = -\hat{k} = \frac{-2\hat{x} + \hat{y} + 2\hat{z}}{3}$$

The azimuth and elevation of such a vector in standard spherical coordinates are:

$$\theta = \cos^{-1}\left(\frac{r_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}}\right) = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.190°$$

$$\phi = \tan^{-1}\left(\frac{r_y}{r_x}\right) = \tan^{-1}\left(\frac{1}{-2}\right) \approx 153.43°$$

Finally, we measure elevation up from the horizon, not down from the z axis, so:

$$\boxed{\text{Elevation angle: } 90° - \theta \approx 41.810° \qquad \text{Azimuth angle: } \phi \approx 153.43°}$$

Problem 3: Mission to Saturn

The planet Saturn is 1.2×10^9 km from Earth at the time a NASA space probe must communicate back to an earth station using a 28 GHz carrier with a minimum received power of -105 dBm. If the satellites transmit amplifier maximum output power is 500 W and the Earth stations receiver dish antenna must be 20 times larger in electromagnetic area than the transmitter antenna (implying 13 dB greater antenna gain), at least how much gain must the satellite dish antenna have? (5 points)

Solution

Start with the link budget equation in a dB scale:

$$P_r = P_t + G_t - 20\log\left(\frac{4\pi f d}{c}\right) + G_r \tag{5}$$

Known relationships:

$$d = 1.2 \times 10^{12}\text{m} \qquad f = 28 \times 10^9\text{Hz} \qquad P_r \geq -105\text{dBm}$$

$$P_t = 10\log\left(\frac{500}{0.001}\right) \approx 57\text{dBm} \qquad G_r = G_t + 13 \qquad c \approx 3 \times 10^8 \text{ m/s}$$

The only unknown is then G_t, plug the known quantities into (5) and solve for G_t:

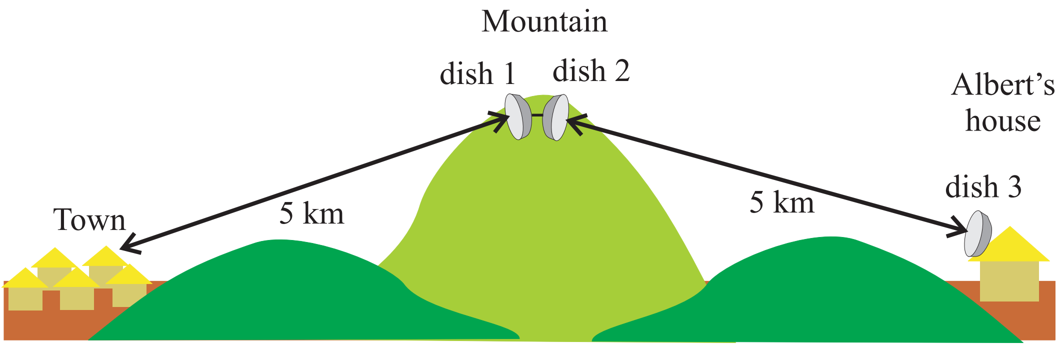
$$-105 \leq 57 + G_t - \underbrace{20\log\left(\frac{4\pi(28 \times 10^9)(1.2 \times 10^{12})}{3 \times 10^8}\right)}_{\approx 303} + G_t + 13$$

$$G_t \geq \frac{-105 - 57 + 303 - 13}{2} = 64\text{dBi}$$

$$\boxed{G_t \geq 64\text{dBi}}$$

Problem 4: Stealing WiFi

Albert lives in a quiet valley in North Georgia where he operates a winery and vineyard. Set apart from civilization, he has no access to cable, DSL, phone lines, or any other wired conduit of internet access. But Albert is a crafty graduate of the Georgia Institute of Technology and devises a clever way to steal WiFi service from the nearby town of Unprotectedlinksville, population 53. This town is 10 kilometers away from Albert, on the other side of a large mountain, and has several unprotected home WiFi servers broadcasting local internet service. His plan is to purchase 3 **identical** dish antennas that operate at 2.45 GHz and arrange them in the following configuration:



With this set-up, a signal will propagate from the town to the first dish in the link, which is pointed toward the town. The received power of this dish is piped directly to another dish which is pointed towards Alberts vineyard. Thus, this pair of dish antennas acts like a passive repeater that does not require any power or maintenance. A third dish is mounted on top of Alberts home, where a minimum value of -95 dBm must be received in order to maintain a wireless internet link on his home computer. Answer the following questions assuming matched and lossless cables. Assume that the antenna gain of the WiFi access point in town is 5 dBi, that the transmit power of this link is 30 dBm, and that both links are essentially free space.

(a) What is the minimum gain in dBi of these antennas to make this system work? (5 points)

(b) If these are ideal circular dishes with 100% efficiency, what is the minimum dish radius based on your answer in part (a)? (5 points)

Solution, Part (a)

There are two link budgets to consider, one from the town to the mountain, and one from the mountain to the vineyard:

$$\text{Town to Mountain: } P_{r1} = P_t + G_t - 20\log\left(\frac{4\pi f d_1}{c}\right) + G_{r1} \tag{6}$$

$$\text{Mountain to House: } P_r = P_{t2} + G_{t2} - 20\log\left(\frac{4\pi f d_2}{c}\right) + G_{r3} \tag{7}$$

The received power at dish 1 is the transmit power of dish 2, since the problem says the pair act as a passive, lossless repeater:

$$P_{r1} = P_{t2} \tag{8}$$

Also, dishes 1, 2, and 3 are identical, so their gains are all the same, call it G:

$$G_{r1} = G_{t2} = G_{r3} = G \tag{9}$$

The approach then is to solve (7) for P_{t2}, then plug this and (6) into (8), subject to (9):

$$P_t + G_t - 20\log\left(\frac{4\pi f d_1}{c}\right) + G = P_r - G + 20\log\left(\frac{4\pi f d_2}{c}\right) - G$$

Solving for G:

$$G = \frac{1}{3}\left(P_r + 20\log\left(\frac{4\pi f d_2}{c}\right) - P_t - G_t + 20\log\left(\frac{4\pi f d_1}{c}\right)\right) \tag{10}$$

The known quantities from the problem statement are:

$$d_1 = d_2 = 5000\text{m} \qquad f = 2.45\times10^9\text{Hz} \qquad P_r \geq -95\text{dBm} \qquad G_t = 5\text{dBi} \qquad P_t = 30\text{dBm} \qquad c \approx 3\times10^8$$

Plugging it all into (10)

$$G \geq \frac{1}{3}\left(-95 + 40\log\left(\frac{4\pi(2.45\times10^9)(5000)}{3\times10^8}\right) - 30 - 5\right) \approx 32.8\text{dBi}$$

So the dish needs at least $\boxed{32.8\text{dBi}}$ of gain.

Solution, Part (b)

Recall the following relationship between antenna aperture and gain in the linear scale:

$$A = \frac{G\lambda^2}{4\pi}$$

If we assume 100% efficiency, the antenna aperture is the physical area. Assuming a circular dish:

$$A = \pi r^2 = \frac{G\lambda^2}{4\pi} \quad \Rightarrow \quad r = \sqrt{\frac{G\lambda^2}{4\pi^2}} = \sqrt{\frac{Gc^2}{4f^2\pi^2}} = \frac{c}{2\pi f}\sqrt{G}$$

Known quantities (remembering to go to the linear gain):

$$c \approx 3\times10^8 \qquad f = 2.45\times10^9\text{Hz} \qquad G = 10^{32.8/10} \approx 1905$$

Which means a circular dish with the following radius gives this amount of gain:

$$\boxed{r = \frac{(3\times10^8)}{2\pi(2.45\times10^9)}\sqrt{1905} \approx 0.85\text{m}}$$