# Review of Fourier Series

ECPE 3614

Lecture 2

Instructor: Gregory D. Durgin







# Outline of Lecture

- Review of Lecture 1
- Calculation of Fourier Coefficients
- Trigonometric vs. Complex FS
- Example 1: Triangular Waveform
- Example 2: Sawtooth Waveform
- Example 3: Rectangular Waveform

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## Review

- Fourier's Theorem: All real periodic functions may be expressed as a linear combination of harmonic sinusoids.
- There are two types of Fourier Series:
  - Trigonometric Fourier Series – uses a series sines and cosines.
  - Complex Fourier Series – uses a series of complex exponentials as oscillators.
  - These series are just two different ways of doing the same thing.





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## The Trigonometric Fourier Series

- Mathematically, the trigonometric Fourier series may be written in this form:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)$$

- Note that the n=0 case for the sine summation is not important since the term drops out to zero

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# Calculating Trigonometric Fourier Series Coefficients

Let's say we want to calculate the $m$th cosine coefficient, $a_m$. How do we go about it? First, start with the definition of the trig. FS:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)$$

Step 1: multiply both sides by the $m$th harmonic cosine as shown below:

$$\cos\left(\frac{2\pi m t}{T}\right) f(t) = \cos\left(\frac{2\pi m t}{T}\right) \left[\sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$$





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# Calculating Trigonometric Fourier Coefficients

Step 2: Integrate both sides over exactly one period of the periodic function. (This is starting to look complicated, but we can simplify in a few steps.)

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi m t}{T}\right) f(t)\, dt =$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi m t}{T}\right) \left[\sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{T}\right)\right] dt$$

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# Calculating Trigonometric Fourier Coefficients

Step 3: The right-side integral may be distributed across each term in the series summation:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi mt}{T}\right)\left[\sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)\right] dt =$$

$$\sum_{n=0}^{\infty} a_n \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi mt}{T}\right) \cos\left(\frac{2\pi nt}{T}\right) dt \; +$$

$$\sum_{n=1}^{\infty} b_n \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi mt}{T}\right) \sin\left(\frac{2\pi nt}{T}\right) dt$$





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# Calculating Trigonometric Fourier Coefficients

Step 4: Take a look and the cos-sin integral:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi mt}{T}\right) \sin\left(\frac{2\pi nt}{T}\right) dt$$

This is a standard definite integral that can be found in many books. Since this is an integration of a harmonic sine wave ($n$ is an integer) and a harmonic cosine wave ($m$ is an integer) over exactly one period, $T$, the integral ***always evaluates to 0***.

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# Calculating Trigonometric Fourier Coefficients

Step 5: There are 3 possible outcomes for the cos-cos definite integral:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi m t}{T}\right)\cos\left(\frac{2\pi n t}{T}\right) dt$$

- *m* and *n* are different: the integral always evaluates to 0.
- *m* and n are the same (but not 0): the integral evaluates to *T/2*.
- *m* and *n* are 0: the integral evaluates to *T*.





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# Calculating Trigonometric Fourier Coefficients

Step 6: Good news! Based on the definite integrations of steps 4 and 5, all but one of the terms will vanish.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi m t}{T}\right) f(t)\, dt =$$

$$\sum_{n=0}^{\infty} a_n \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi m t}{T}\right)\cos\left(\frac{2\pi n t}{T}\right) dt \; +$$

always 0 except for the case *m*=*n*.

$$\sum_{n=1}^{\infty} b_n \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi m t}{T}\right)\sin\left(\frac{2\pi n t}{T}\right) dt$$

always 0

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# Calculating Trigonometric Fourier Coefficients

Step 7: For the case of m>0, the results of our hard efforts produces an equation for $a_m$:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{2\pi m t}{T}\right) f(t)\,dt = a_m \frac{T}{2}$$

Just switch the *m* to an *n* (to keep notation consistent) and rearrange the equation to solve for $a_n$. Now we have the formula for the *n*th cosine Fourier coefficient!

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos\left(\frac{2\pi n t}{T}\right) dt$$







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# Calculating Trigonometric Fourier Coefficients

Step 8: One last step – for the case of m=0, the results of our hard efforts produces an equation for $a_0$:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt = a_0 T$$

Rearranging, the equation for $a_0$ becomes…

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt$$

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# Calculating the Trigonometric Fourier Coefficients…

The full set of trigonometric Fourier coefficients is shown on the right. The derivation for the $b_n$ is identical to the $a_n$, except that a sine term is used in Step 1. Note: the limits of integration for these formulas are somewhat arbitrary. Since the integrands are periodic functions, any integration length that spans exactly one period, $T$, will produce the correct result.

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2\pi n t}{T}\right)\,dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi n t}{T}\right)\,dt$$





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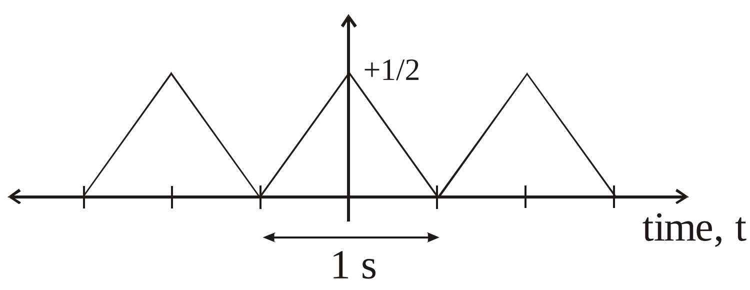
# Example of Trig FS: A Triangular Waveform

- Blackboard Example: Find the trigonometric Fourier coefficients for this triangular waveform with period of 1s.



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# The Complex Fourier Series

- Below is the complex Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi nt}{T}\right)$$

- Key points about the complex series:
  - unlike $a_n$ and $b_n$, $c_n$ may be a complex number.
  - summation limits are from –infinity to +infinity.
  - if f(t) is to be a real-valued function, the complex Fourier coefficients must have special properties (see Homework 1, problem 1).





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# Calculating The Complex Fourier Coefficients

Start with the definition of the complex Fourier Coefficients:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi nt}{T}\right)$$

Step 1: To calculate the *m*th coefficient, multiply both sides of the equation by the –*m*th complex exponential harmonic:

$$\exp\left(-j\frac{2\pi mt}{T}\right) f(t) = \exp\left(-j\frac{2\pi mt}{T}\right) \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi nt}{T}\right)$$

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# Calculating The Complex Fourier Coefficients

Step 2: Integrate both sides over an entire period, *T*.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \exp\left(-j\frac{2\pi mt}{T}\right) f(t)\,dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[\exp\left(-j\frac{2\pi mt}{T}\right) \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi nt}{T}\right)\right] dt$$

Note: The imaginary *j* does not change how you perform integration. Just take it along for the ride, as if it were a constant.





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# Calculating The Complex Fourier Coefficients

Step 3: Distribute the integral to each term inside the summation. This step is valid because integration is *linear*.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \exp\left(-j\frac{2\pi mt}{T}\right) f(t)\,dt =$$

$$\sum_{n=-\infty}^{\infty} c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} \exp\left(-j\frac{2\pi mt}{T}\right) \exp\left(j\frac{2\pi nt}{T}\right) dt =$$

$$\sum_{n=-\infty}^{\infty} c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} \exp\left(j\frac{2\pi(n-m)t}{T}\right) dt$$

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# Calculating The Complex Fourier Coefficients

Step 4: Study the following integral:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \exp\left(j\frac{2\pi(n-m)t}{T}\right)\,dt$$

0 unless $m=n$

Since this is an integration of complex harmonic sinusoids (the value *n-m* is an integer) over exactly one period, the result is only non-zero for $m=n$. For $m=n$, the integral evaluates to *T*.





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# Calculating The Complex Fourier Coefficients

Step 5: After all of the zero terms are removed, the final result is given below:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \exp\left(-j\frac{2\pi m t}{T}\right) f(t)\,dt = c_m T$$

Swapping the letter m for n (for consistency of notation) and re-arranging provides us the formula for the complex Fourier coefficients:

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\exp\left(-j\frac{2\pi n t}{T}\right)\,dt$$

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# The Complex Fourier Series

Below is the integral for calculating complex Fourier coefficients. The drawback to this method is the integration of the complex exponent. This integration often produces a complex coefficient for $c_n$. One nice advantage: only one formula as opposed to the 3 equations required for the trigonometric series.

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \exp\left(-j\frac{2\pi n t}{T}\right) dt$$





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# To Convert from Complex FS to Trig FS Coefficients…

…just make the following substitutions. This can save a lot of computation!

$$a_0 = c_0$$

$$a_n = c_n + c_{-n}$$

$$b_n = j(c_n - c_{-n})$$

Remember: the complex Fourier series is just a different way to keep track of the spectral decomposition.

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# To Convert from Trig FS to Complex FS Coefficients…

…just make the following substitutions.

$$c_0 = a_0$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

The set of harmonic sinusoids (both complex exponential *and* trigonometric) are said to be a ***basis*** that ***spans*** the set of real periodic functions.





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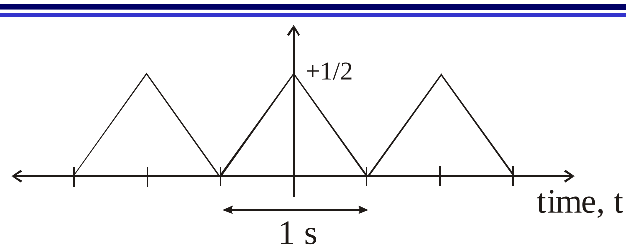
# Ex. 1: Triangular Waveform




Trig FS coefficients:

$$a_n = \begin{cases} \frac{1}{4} & \text{for } n = 0 \\ \frac{2}{\pi^2 n^2} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

$$b_n = 0$$

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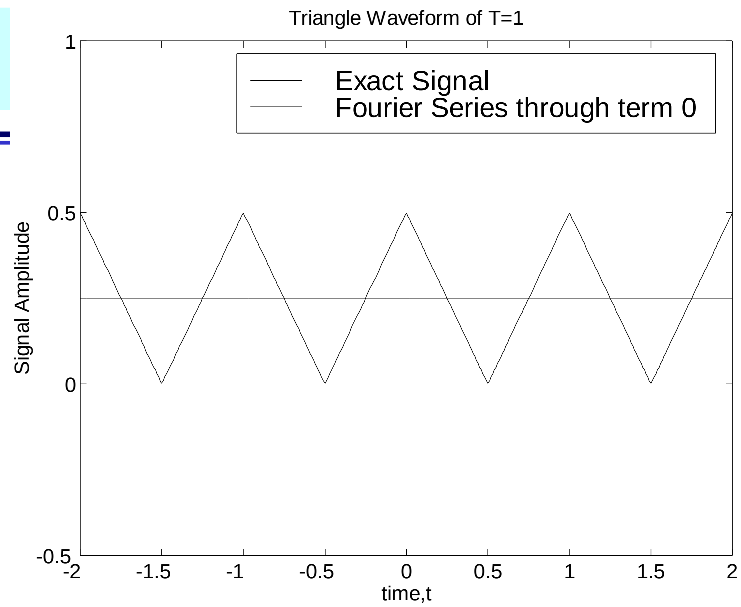


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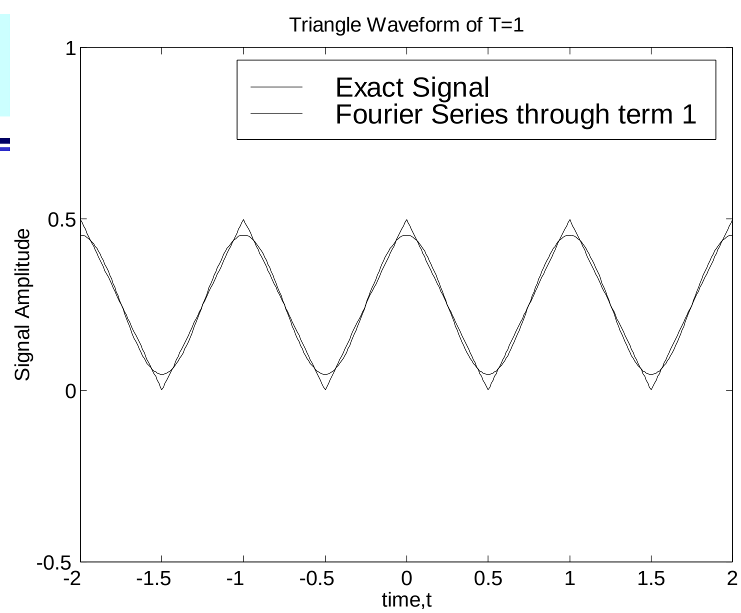


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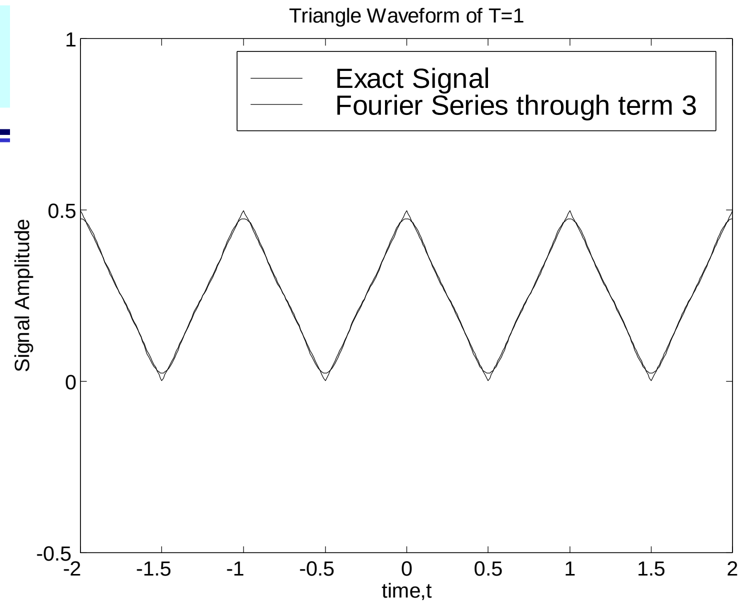


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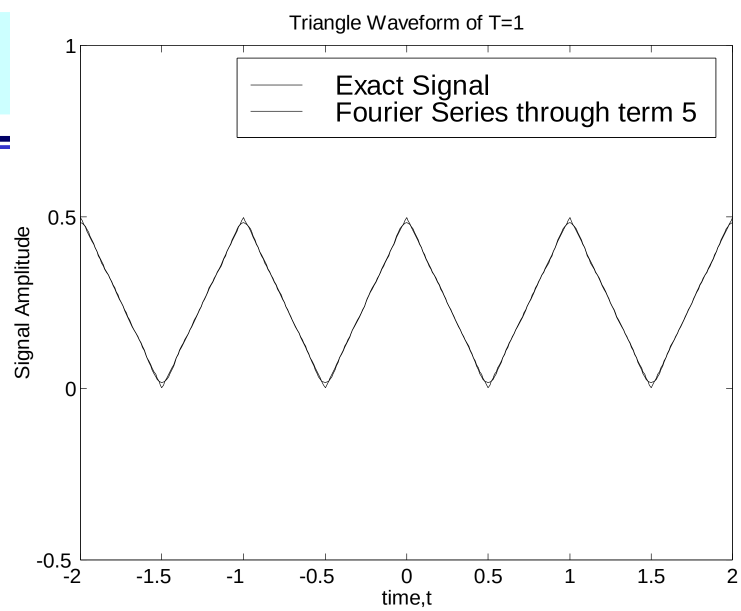


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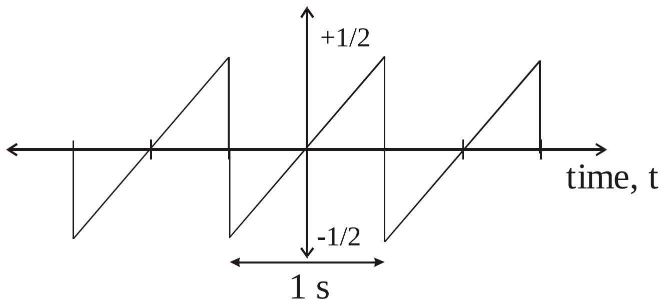
# Ex. 2: The Sawtooth Waveform




Trig FS coefficients:

$$a_n = 0$$

$$b_n = \frac{(-1)^{n+1}}{\pi n}$$

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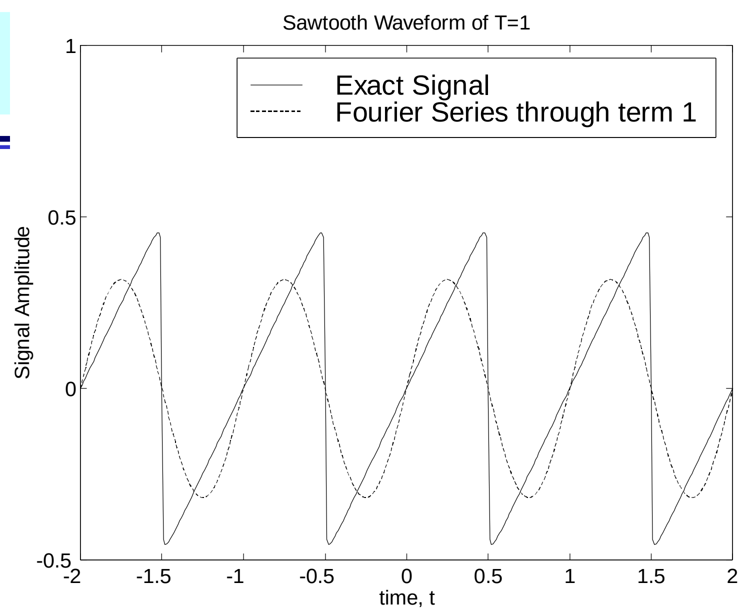


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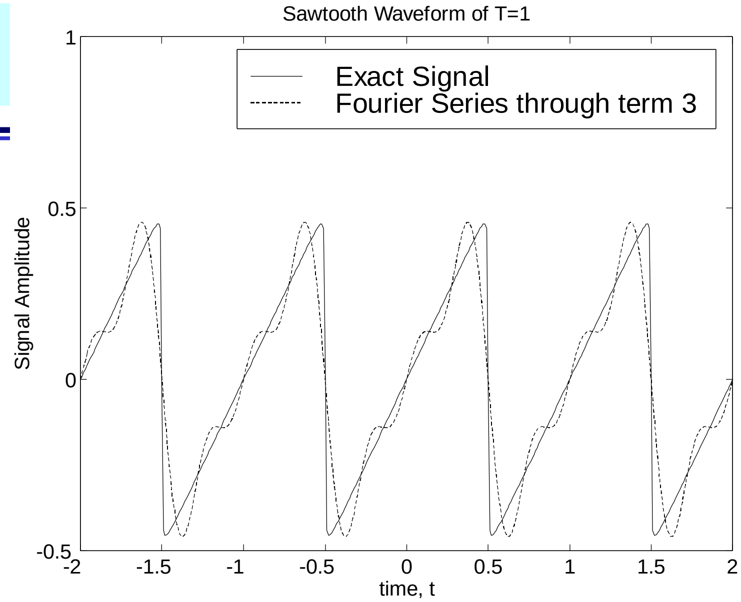


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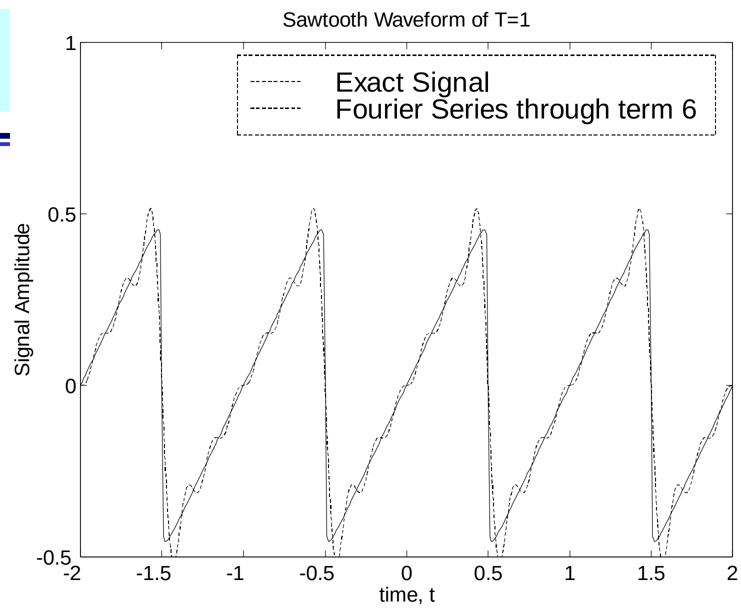


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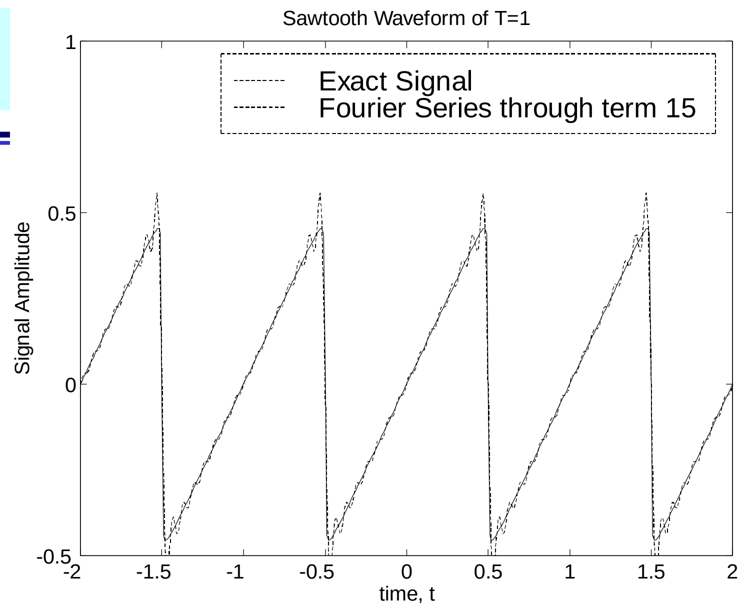






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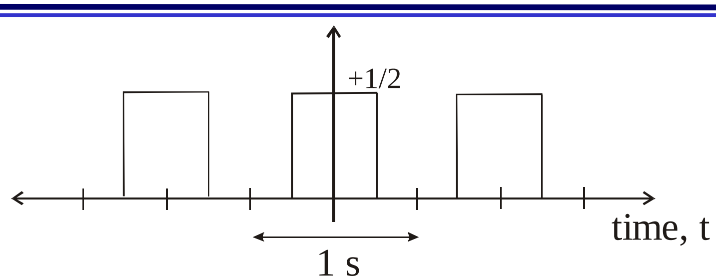
# Ex. 3: The Rectangular Waveform




Trig FS coefficients:

$$a_n = \begin{cases} \frac{1}{4} & \text{for } n = 0 \\ \frac{(-1)^{\frac{n+1}{2}}}{\pi n} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

$$b_n = 0$$

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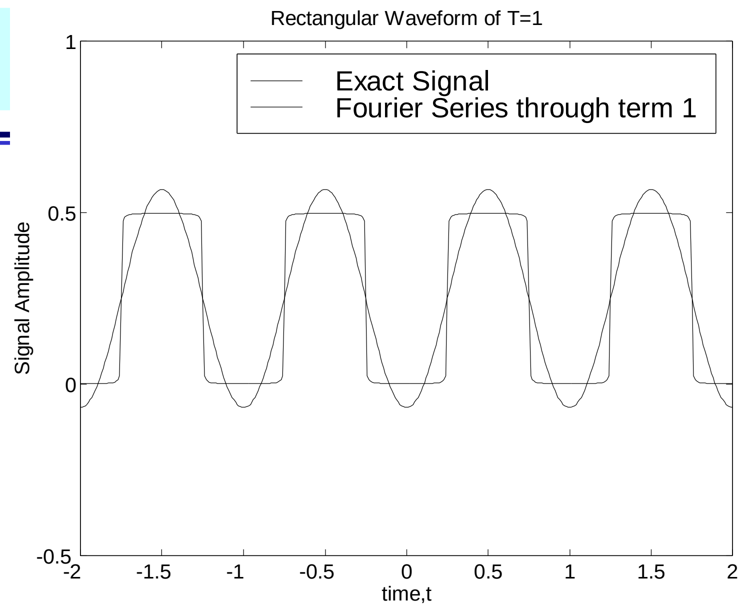


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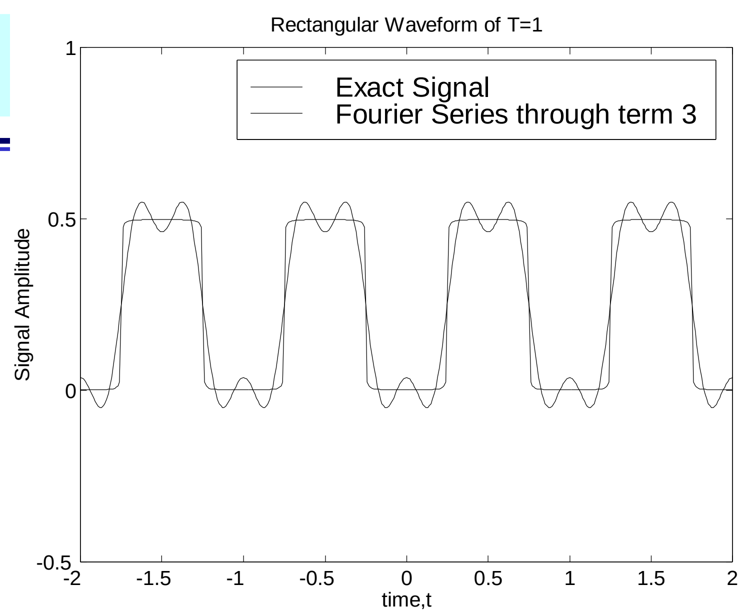


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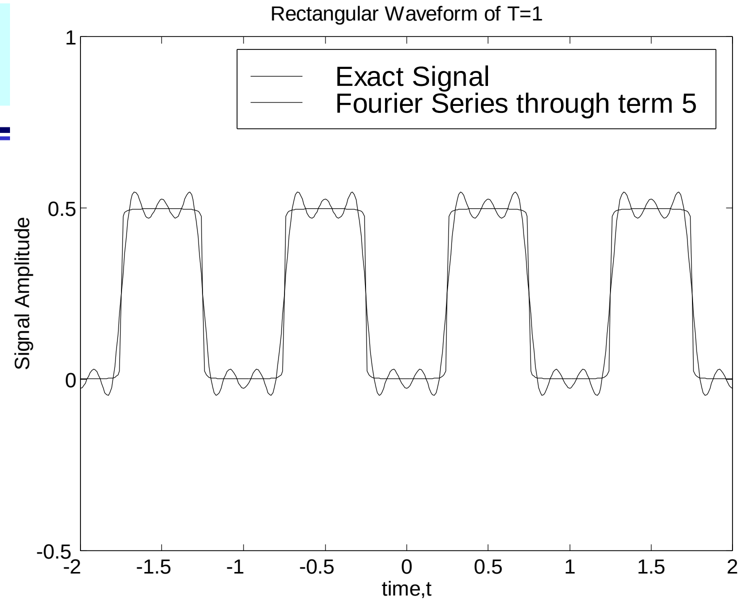
Rectangular Waveform of T=1
Exact Signal
Fourier Series through term 5
Signal Amplitude
time,t
1
0.5
0
-0.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2







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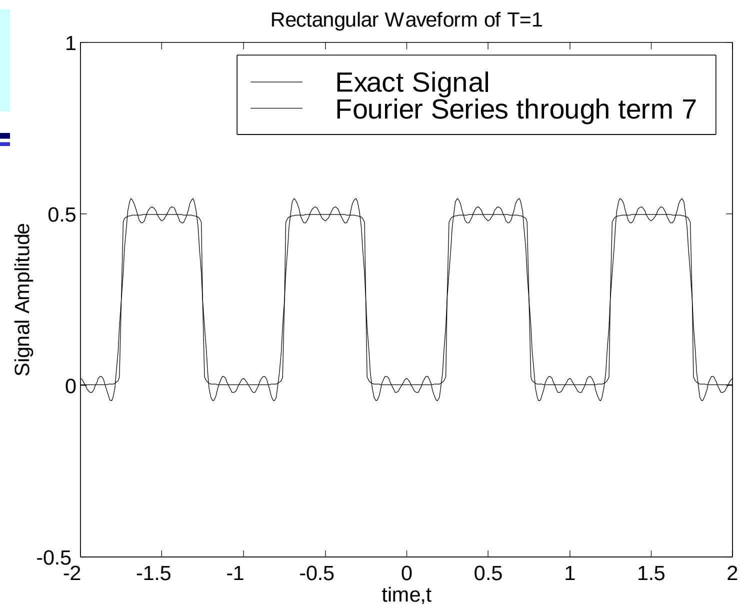
Rectangular Waveform of T=1
Exact Signal
Fourier Series through term 7
Signal Amplitude
time,t
1
0.5
0
-0.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

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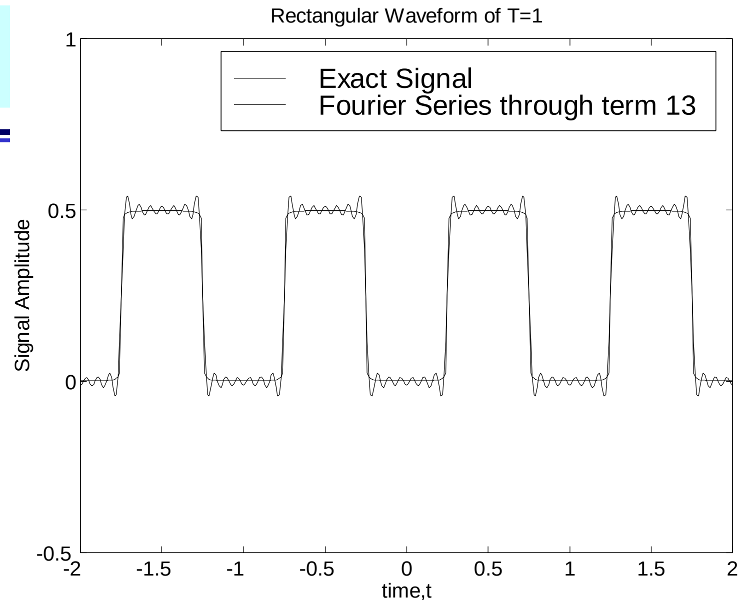


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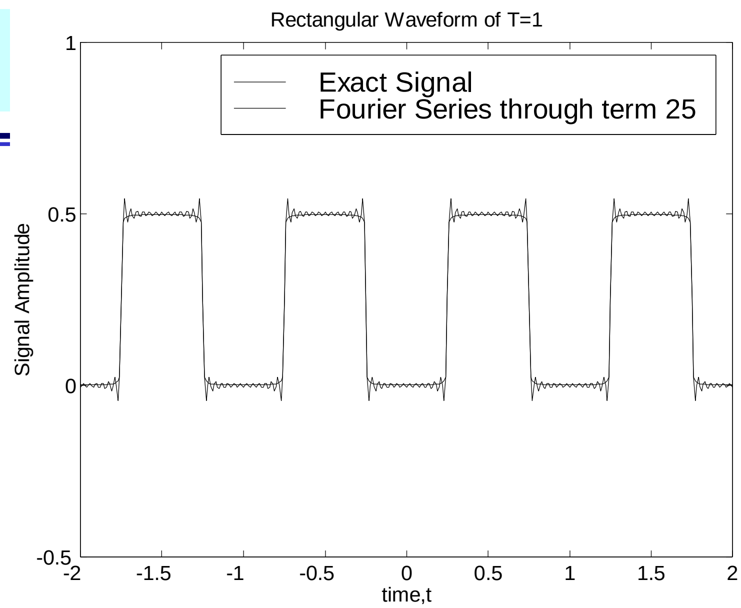


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# Key Points of Lecture

- Basic Form of the Fourier Series
- Calculate Trigonometric Fourier Coefficients
- Calculate Complex Fourier Coefficients
- Convert Between Complex and Trig Fourier Coefficients
- Examples of how truncated Fourier Series approximate real signals with increasing accuracy