The Solution

The first thing we want to determine is the latitude of the crash site. We know that the SARSAT satellite, orbiting the Earth along the $-76°$ longitude line, crosses the equator at the time 0 seconds. From that point the satellite listens to the emitted signal from an EPIRB of the crashed plane and analyzes its frequency with respect to time. Figure 1 shows the result.

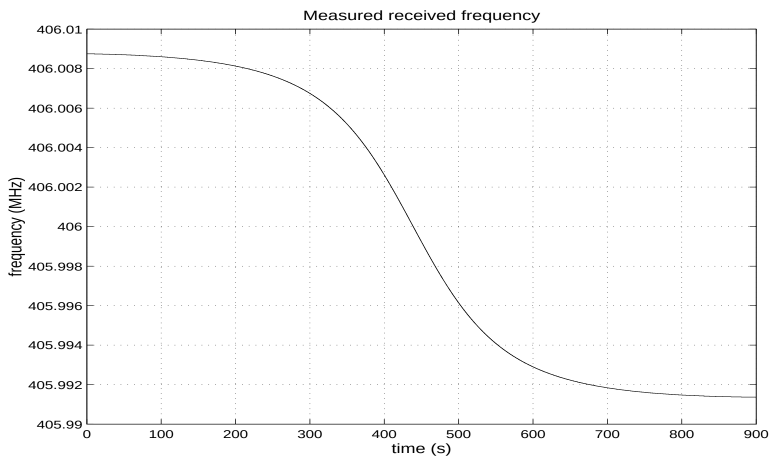

Figure 1: The received frequency of the EPIRB signal with respect to time.

At the point in time when the satellite reaches the latitude of the crash site, the velocity vector of the satellite and the wave vector of the signal emitted from the EPIRB will be exactly perpendicular to each other. Consequently, there will be no Doppler shift in the received signal at the satellite and the received frequency will be equal to the actual transmit frequency of the signal.

The received frequency changes fastest around the point in time when the Doppler shift is zero, hence by determining when that happens we know when the satellite crosses the latitude of the crash site. Now, let us define $f_r(t)$ as the received frequency with respect to time. The Doppler shift in the received signal is zero when the time derivative of $f_r(t)$, $df_r(t)/dt$, takes the maximum value. We analyzed $df_r(t)/dt$ (see Figure 2) and determined that it reaches the maximum at the time 438 seconds. The received frequency at that time is 406000070 Hz and equals the transmit frequency.

Since we set the time to 0 when the satellite crosses the equator, the latitude of the crash site equals the angle that the satellite traverses in 438 seconds. The angular velocity of the satellite equals

$$\omega_s = \frac{v_s}{r} = \frac{v_s}{R_e + h} = \frac{7.3336 \text{ km/s}}{6380 \text{ km} + 850 \text{ km}} = 1.014329 \cdot 10^{-3} \text{ rad/s},$$

where v_s, R_e and h designate the velocity of the satellite, the radius of the Earth and the altitude of the satellite from the Earth's surface, respectively.

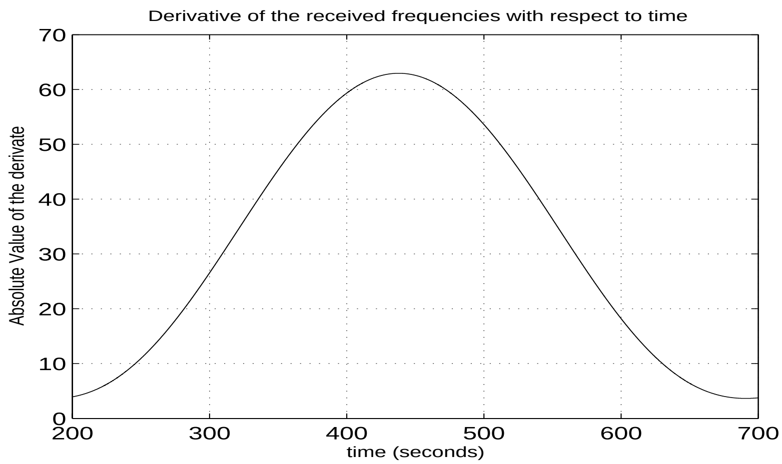

Figure 2: The derivative of the received frequency with respect to time.

The latitude of the crash site, α, is thus

$$L = \omega t = 1.014329 \cdot 10^{-3} \text{ rad/s } \cdot 438s = 0.444276 \text{ rad } = 25.455°.$$

The calculation of longitude is considerably more difficult. While various approaches can be applied to solve this problem, we chose one that in our opinion is elegant and provides a very good estimate of the actual longitude of the crash site.

The basic idea is to calculate the range of the crash site from the satellite when the satellite crosses the latitude of the crash site. We found a research paper [1] that describes how that range can be derived by carefully defining the geometry of the problem and solving a set of equations. Figure 3 shows the geometry as defined by the authors in [1].

The point C on the bottom of the figure represents the center of the Earth, while the point A represents the satellite as it crosses the latitude of the crash site. Since the satellite is riding on a circular orbit with the altitude h, the length of CA is $R_e + h$. The point B represents an arbitrary location on the satellite's orbit, hence the length of CB is equal to $CA = R_e + h$. The point O represents the center of the coordinate system and was chosen to be at the intersection of CA and a line that is perpendicular to CA and goes through the point E that represents the crash site. The length of OE, denoted as Z_0, can hence be interpreted as the distance of the crash site to the line connecting the center of the Earth and the satellite when at A. When the satellite is at A its subpoint is located on CA somewhere above O, since R_p is inevitably less than R_e. The angle that the satellite traverses when traveling between A and B is denoted as Θ_b and α_b denotes the angle between the velocity vector of the satellite when at B and the wave vector of the emitted EPIRB signal as seen by the satellite at B. We will need some of the remaining notations from Figure 3, but we do not discuss them here since they are quite self-explanatory.

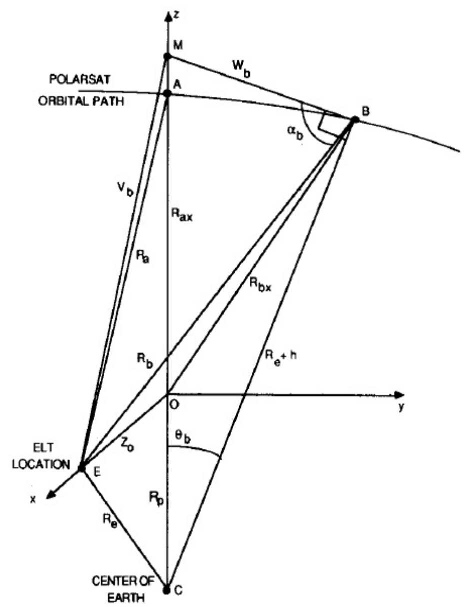

Figure 3: The geometry of the system (from [1]).

The first objective is to calculate the distance between the satellite when at A and the crash site. With this in mind we start by stating the following relationships from Figure 3:

$$
\begin{aligned}
R_a^2 &= R_{ax}^2 + Z_0^2 && (1)\\
R_e^2 &= R_p^2 + Z_0^2 && (2)\\
R_{ax} &= R_e + h - R_p && (3)\\
R_b^2 &= R_{bx}^2 + Z_0^2 && (4)\\
R_{bx}^2 &= (R_e + h)^2 + R_p^2 - 2R_p(R_e + h)\cos\Theta_b. && (5)
\end{aligned}
$$

By plugging (2) and (5) into (4) we get

$$R_b^2 = (R_e + h)^2 + R_e^2 - 2R_p(R_e + h)\cos\Theta_b. \tag{6}$$

It is shown in [1] that

$$R_p = R_b \frac{\cos\alpha_b}{\sin\Theta_b} \tag{7}$$

and by using this result we can expand (6) to

$$R_b^2 = (R_e + h)^2 + R_e^2 - 2R_b(R_e + h)\frac{\cos\alpha_b}{\tan\Theta_b}, \tag{8}$$

which is a quadratic equation for R_b with the solution

$$R_b = -(R_e + h)\frac{\cos\alpha_b}{\tan\Theta_b} + \sqrt{\left(\left((R_e + h)\frac{\cos\alpha_b}{\tan\Theta_b}\right)^2 + R_e^2 + (R_e + h)^2\right)}. \tag{9}$$

If we know the frequency of the EPIRB signal as seen by the satellite at B, denoted by f_{rB}, α_b is

$$\alpha_b = \cos^{-1}\left(\frac{f_{rB} - f_t}{f_d}\right),$$

where f_t is the transmit frequency, $f_d = v_s/\lambda_0$ is the Doppler shift along the path of travel of the satellite, and λ_0 is the wavelength of the EPIRB signal. We already calculated the angular velocity of the satellite ω_s, thus Θ_b is given by $\omega_s T$, where T is the time the satellite needs to travel from A to B.

Finally, by using (1), (2), and (3) we get

$$R_a^2 = (R_e + h - R_p)^2 + R_e^2 - R_p^2$$

or

$$R_a = \sqrt{\left((R_e + h)^2 - 2R_p(R_e + h) + R_e^2\right)}.$$

Now, after all the rigorous math, we know how to calculate the range between the crash site and the satellite when at A, that is when it crosses that latitude line of the crash site. It remains to be shown how one can calculate the longitude with this information.

Let us start by giving some insight into the geometry of the new problem. Figure 4 shows the cross section of the planet Earth along the $-76°$ longitude line. The point A in this figure is equivalent to the point A in Figure 3. Thus, when the satellite is at point A it crosses the latitude of the crash site.

If the range between the satellite at point A and the crash site is R_a, the possible locations of the crash site are at points where a sphere with the center in A and radius R_a intersects the surface of the planet Earth. More exactly, since we know the latitude of the crash site is 25.455°, the location of the crash site must be at the intersection of the above mentioned sphere and the 25.455° latitude line on the Earth's surface.

Let us now define the three dimensional coordinate system like shown in Figure 5. Clearly, the 25.455° latitude line is a circle that lies in the $x-y$ plane at $z = 0$ and is defined as

$$x^2 + y^2 = R_{e1}^2, \tag{10}$$

where R_{e1} is the radius of the Earth along the 25.455° latitude line and is equal to $R_e \cdot \sin(90° - 25.455°)$.
The sphere, in this coordinate system, is defined as

$$(x - (R_{e1} + h_x))^2 + y^2 + (z - h_z)^2 = R_a^2, \tag{11}$$

where $h_x = h \cdot \cos(25.455°)$ and $h_z = h \cdot \sin(25.455°)$. We determine where these two curves intersect by solving the system of equations defined by (10) and (11) at $z = 0$. If we subtract (11) from (10) we immediately get the solution

$$x_s = \frac{R_{e1}^2 - R_a^2 + h_z^2 + (R_{e1} + h_x)^2}{2(R_{e1} + h_x)}.$$

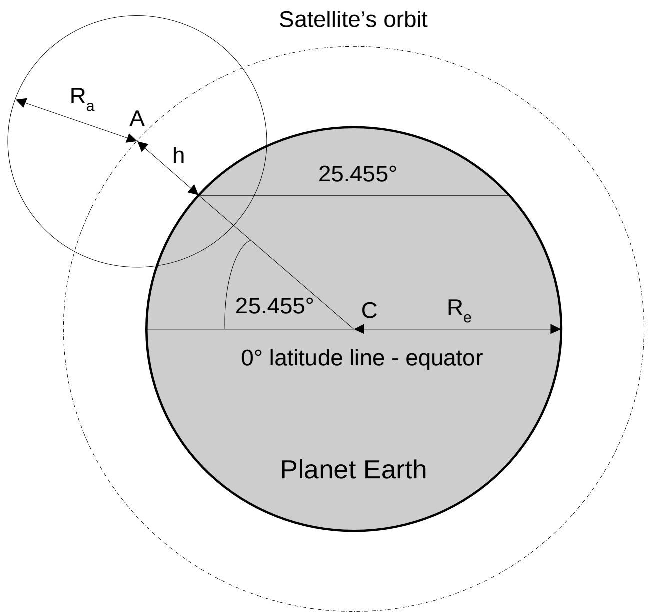

Figure 4: The cross section of the Earth along the $-76°$ longitude line.

Figure 6 shows what we have just done. The sphere at $z = 0$ is a circle with an origin at $x = R_{e1} + h_x$ and radius h_x. Thus, we have determined where this circle and (10) intersect.

Actually, we see that they intersect at two distinct locations and both of them could potentially be a location of the crash site. One has to check both and decide which makes more sense.

The offset of the crash site with respect to the satellite's longitude is represented by the angle β in Figure 6. To get β we first determine the y-coordinate of the two intersections by plugging x_s in (10). Now β is given by

$$\beta = \tan^{-1} \frac{|y_s|}{x_s}$$

and the two possible longitudes of the crash site are

$$l_1 = -76° + \beta$$

and

$$l_2 = -76° - \beta.$$

In our calculations we choose the point B (see Figure 3) to be at time 704.9 seconds, that is 266 seconds after the satellite crosses point A. At point B the satellite receives the frequency 405991810 Hz.

With these data β results to 3.155° and the two possible locations are $(25.455°, -72.845°)$ and $(25.455°, -79.155°)$. There are no islands in the prox-

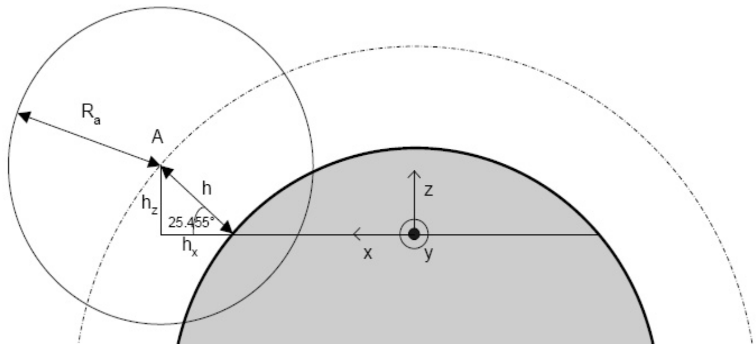

Figure 5: The coordinate system.

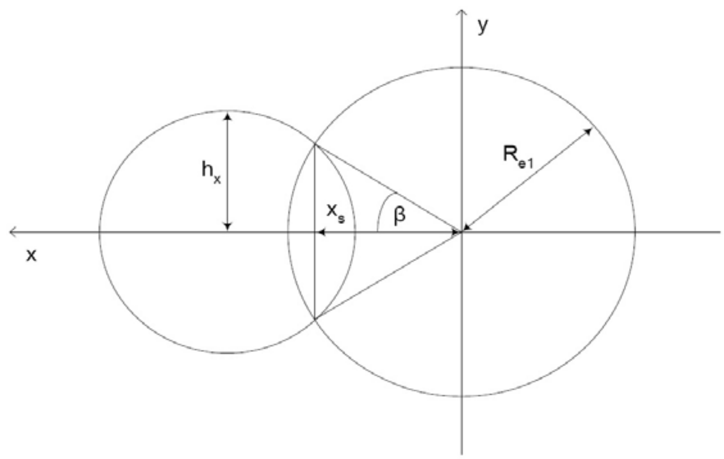

Figure 6: The longitude.

imity of the first location, while the second one is just beside the island **Bimini** in the Bahamas.

References

[1] G. Vrckovnik and C. R. Carter, "A Novel Approach for the SARSAT System," *IEEE Trans. Aerospace and Electronic Systems.*, vol. 27, pp. 290—301, March 1991.